Further extensions of Clifford circuits and their classical simulation complexities

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Motivating questions

- What is the relationship between classical and quantum computational power?
- What are the computational benefits of various kinds of quantum resources?

Restricted models of quantum computation and their classical simulation complexities

- A restricted model of quantum computation is a one that has specified limited quantum ingredients.
- Classical simulation complexity of a restricted model: how hard it is to classically simulate the model.
  - Computational hardness is notoriously difficult to prove, so it is popular to resort to only providing evidence of hardness.
- Studying the classical simulation complexity of restricted models can help us identify which ingredients are an essential 'resource' for quantum computational power.
- Example: Extended Clifford circuits, which straddle the boundary between classical and quantum computational power.

Clifford circuits

- The Pauli group is the set of operators of the form $P = i^k P_1 \otimes \ldots \otimes P_n$, where $k = 0, 1, 2, 3$ and each $P_i \in \{1, X, Y, Z\}$ is a Pauli matrix.
- The n-qubit *Clifford group* $\mathcal{C}_n$ is the normalizer of the Pauli group $\mathcal{P}_n$ in the n-qubit unitary group $\mathcal{U}_n$, i.e. $\mathcal{C}_n = \{U \in \mathcal{U}_n | U\mathcal{P}_n U^\dagger = \mathcal{P}_n\}$. Elements of the Clifford group are called *Clifford operations*.
- Claim: An operator C is a Clifford operation iff it can be implemented by a circuit consisting of the following gates (called the *basic Clifford gates*):
  - Hadamard gate $H = 1/\sqrt{2}(X + Z)$
  - phase gate $S = \mathrm{diag}(1, i)$
  - CNOT gate $CX_{ab} = |0\rangle\langle 0|_a \otimes I_b + |1\rangle\langle 1|_a \otimes X_b$
- A *Clifford circuit* (or *stabilizer circuit*) is one that consists of the basic Clifford gates and single-qubit intermediate measurement gates in the computational basis.

Clifford circuits

- Numerous applications in quantum error correction, measurement-based quantum computing, etc.
- Rich enough to encompass many ‘quantum’ features like quantum teleportation and entanglement.

Gottesman-Knill Theorem

Theorem (Gottesman-Knill)

Clifford circuits can be efficiently simulated on a classical computer.

Gottesman-Knill Theorem

Theorem (Gottesman-Knill)

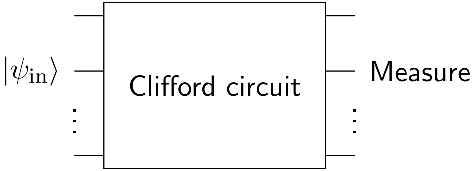
Clifford circuits can be efficiently simulated on a classical computer.

- is known to be true only in a suitably restricted setting. It depends on
    - notion of efficient classical simulation
    - ingredients of the Clifford circuit
- Circuits with different ingredients are called *extended Clifford circuits*.

Goals

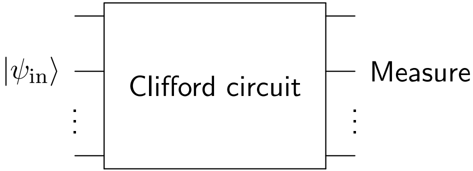
- Discuss extensions of Clifford circuits and clarify the different notions of classical simulation of quantum computation
- Determine which extended Clifford circuits are efficiently classically simulable
- Provide evidence that particular extended Clifford circuits are not efficiently classically simulable (based on plausible complexity assumptions).

Three types of ingredients of Clifford circuits



1. Inputs: IN(BITS) vs IN(PROD)
   - IN(BITS): computational basis inputs, i.e. $|\psi_{\text{in}}\rangle = |x_1, \ldots, x_n\rangle$, where $x_i \in \{0, 1\}$.
   - IN(PROD): product state inputs: i.e. $|\psi_{\text{in}}\rangle = |\alpha_1\rangle \otimes \ldots \otimes |\alpha_n\rangle$, where $\alpha_i \in \mathbb{C}^2$.

Three types of ingredients of Clifford circuits

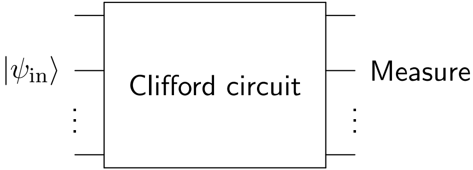


2. Intermediate measurements: ADAPTIVE vs NONADAPTIVE
   - ADAPTIVE: Input states transform as follows:

$$
\begin{aligned}
|\psi\rangle \quad \rightarrow \quad & C_K(x_1,\ldots,x_K)M_{i_K(x_1,\ldots,x_{K-1})}(x_K)\ldots \\
& \qquad C_2(x_1,x_2)M_{i_2(x_1)}(x_2)C_1(x_1)M_{i_1}(x_1)C_0|\psi\rangle
\end{aligned}
$$

   - NONADAPTIVE: same as above, except that C_i's and i_k's do not depend on $x_1,\ldots,x_K$.

Three types of ingredients of Clifford circuits



3. Outputs: OUT(BITS) vs OUT(PROD)
   - OUT(BITS): computational basis measurements.
   - OUT(PROD): arbitrary single-qubit measurements.

Notions of classical simulation

Informal definitions:

- Strong (called STR) simulation of a quantum circuit: a classical algorithm that calculates the probabilities of any subset of outcomes of the quantum circuit.
- Weak (called WEAK) simulation of a quantum circuit: an classical algorithm that samples from the same distribution as the quantum circuit.
- Strong-$f(n)$ (called STR($f(n)$)) simulation of a quantum circuit: like strong simulation, except that the size of the subset to be simulated is equal to $f(n)$.
- Weak-$f(n)$ (called WEAK($f(n)$)) simulation of a quantum circuit: like weak simulation, except that the size of the subset to be simulated is equal to $f(n)$.

Notions of classical simulation

Let $\mathcal{C}_\nu$ be a set of Clifford circuits.
Let $p_T^I(y) =$ probability that when the registers (indexed by the set I) of the quantum circuit T are measured, the outcomes y are observed.

Definition (strong simulation)

A STR-simulation of $\mathcal{C}_\nu$ is a deterministic classical algorithm with
Input: $\langle T, I, y\rangle$, where

- $T \in \mathcal{C}_\nu$ is a Clifford circuit on n qubits.
- $I = \{i_1, \ldots, i_{|I|}\} \subseteq [n]$
- $y \in \{0, 1\}^{|I|}$

Output: $p_T^I(y)$ (up to exponential precision)

Notions of classical simulation

Let $\mathcal{C}_\nu$ be a set of Clifford circuits.
Let $p^I_T(y) =$ probability that when the registers (indexed by the set I) of the quantum circuit T are measured, the outcomes y are observed.

Definition (strong-$f(n)$ simulation)

A STR($f(n)$)-simulation of $\mathcal{C}_\nu$ is a deterministic classical algorithm with
Input: $\langle T, I, y\rangle$, where

- $T \in \mathcal{C}_\nu$ is a Clifford circuit on n qubits.
- $I = \{i_1, \ldots, i_{f(n)}\} \subseteq [n]$ is of size $f(n)$
- $y \in \{0,1\}^{f(n)}$

Output: $p^I_T(y)$ (up to exponential precision)

Notions of classical simulation

Let $\mathcal{C}_\nu$ be a set of Clifford circuits.
Let $p_T^I(y) =$ probability that when the registers (indexed by the set I) of the quantum circuit T are measured, the outcomes y are observed.

Definition (weak simulation)

A WEAK-simulation of $\mathcal{C}_\nu$ is a randomized classical algorithm with
Input: $\langle T, I \rangle$, where

- $T \in \mathcal{C}_\nu$ is a Clifford circuit on n qubits.
- $I = \{i_1, \ldots, i_{|I|}\} \subseteq [n]$

Output: y with probability $p_T^I(y)$

Notions of classical simulation

Let $\mathcal{C}_\nu$ be a set of Clifford circuits.
Let $p_T^I(y)$ = probability that when the registers (indexed by the set I) of the quantum circuit T are measured, the outcomes y are observed.

Definition (weak-$f(n)$ simulation)

A WEAK($f(n)$)-simulation of $\mathcal{C}_\nu$ is a randomized classical algorithm with
Input: $\langle T, I \rangle$, where

- $T \in \mathcal{C}_\nu$ is a Clifford circuit on n qubits.
- $I = \{i_1, \ldots, i_{f(n)}\} \subseteq [n]$ is of size $f(n)$

Output: y with probability $p_T^I(y)$.

Relationships between notions of classical simulation

Restrict our attention to the cases where $f(n) = 1$ or n.

$$\begin{tikzcd}
& \mathrm{STR} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{WEAK} = \mathrm{WEAK}(n) \arrow[dr] & \mathrm{STR}(1) \arrow[d] & \mathrm{STR}(n) \arrow[dl] \\
& \mathrm{WEAK}(1) &
\end{tikzcd}$$

Figure: $A \to B$ means that an efficient A-simulation of a computational task implies that there is an efficient B-simulation for the same task.

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)					
		IN (PROD)					
	ADAPT	IN (BITS)					
		IN (PROD)					
OUT (PROD)	NON-ADAPT	IN (BITS)					
		IN (PROD)					
	ADAPT	IN (BITS)					
		IN (PROD)					

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)					
		IN (PROD)					
	ADAPT	IN (BITS)		$\boxed{\text{P}}$ (GK)			
		IN (PROD)					
OUT (PROD)	NON-ADAPT	IN (BITS)					
		IN (PROD)					
	ADAPT	IN (BITS)					
		IN (PROD)					

GK = Gottesman-Knill [Gottesman '98]. P means efficiently classically simulable.

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)			
		IN (PROD)					
	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (GK)			
		IN (PROD)					
OUT (PROD)	NON-ADAPT	IN (BITS)					
		IN (PROD)					
	ADAPT	IN (BITS)					
		IN (PROD)					

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)			$\boxed{\text{P}}$ (JV4)
		IN (PROD)					
	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (GK/JV5)			
		IN (PROD)					
OUT (PROD)	NON-ADAPT	IN (BITS)					
		IN (PROD)			$\boxed{\text{P}}$ (Thm 5)		
	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)				
		IN (PROD)					

JV = Jozsa and Van den Nest [Jozsa and Van den Nest '14]

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
		IN (PROD)	P (v)		P (JV1)		
	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)			
		IN (PROD)					
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)		P (xiii)		
		IN (PROD)	P (xv)		$\boxed{\text{P}}$ (Thm 5)		
	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)				
		IN (PROD)					

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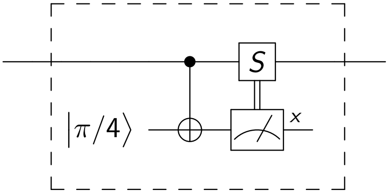
Magic states

- Clifford + T gate is universal for quantum computation.
- The T gate can be simulated by the following gadget:



where $\left|\pi/4\right\rangle = \frac{1}{2}(|0\rangle + e^{i\pi/4}|1\rangle)$.

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
OUT (BITS)	NON-ADAPT	IN (PROD)	P (v)		P (JV1)		
OUT (BITS)	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)			
OUT (BITS)	ADAPT	IN (PROD)	$\boxed{\text{QC}}$ (JV3)				
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)		P (xiii)		
OUT (PROD)	NON-ADAPT	IN (PROD)	P (xv)		$\boxed{\text{P}}$ (Thm 5)		
OUT (PROD)	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)				
OUT (PROD)	ADAPT	IN (PROD)					

QC means universal for quantum computation

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
OUT (BITS)	NON-ADAPT	IN (PROD)	P (v)		P (JV1)		
OUT (BITS)	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)			
OUT (BITS)	ADAPT	IN (PROD)	$\boxed{\text{QC}}$ (JV3)	QC (viii)			
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)		P (xiii)		
OUT (PROD)	NON-ADAPT	IN (PROD)	P (xv)		$\boxed{\text{P}}$ (Thm 5)		
OUT (PROD)	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)				
OUT (PROD)	ADAPT	IN (PROD)	QC (xxiii)	QC (xxiv)			

QC means universal for quantum computation

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
		IN (PROD)	P (v)	$\boxed{\text{PH}}$ (JV7)	P (JV1)		
	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)			
		IN (PROD)	$\boxed{\text{QC}}$ (JV3)	QC (viii)			
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)	$\boxed{\text{PH}}$ (Thm 3)	P (xiii)		
		IN (PROD)	P (xv)		$\boxed{\text{P}}$ (Thm 5)		
	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)				
		IN (PROD)	QC (xxiii)	QC (xxiv)			

PH means that if the problem is efficiently classical simulable, then the polynomial hierarchy collapses.

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
		IN (PROD)	P (v)	$\boxed{\text{PH}}$ (JV7)	P (JV1)		
	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)			
		IN (PROD)	$\boxed{\text{QC}}$ (JV3)	QC (viii)			
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)	$\boxed{\text{PH}}$ (Thm 3)	P (xiii)		
		IN (PROD)	P (xv)	PH (xvi)	$\boxed{\text{P}}$ (Thm 5)		
	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)	PH (xix)			
		IN (PROD)	QC (xxiii)	QC (xxiv)			

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OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
		IN (PROD)	P (v)	$\boxed{\text{PH}}$ (JV7)	P (JV1)	$\boxed{\#\text{P}}$ (Thm 1)	
	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)	$\boxed{\#\text{P}}$ (JV2)	$\boxed{\#\text{P}}$ (Thm 2)	
		IN (PROD)	$\boxed{\text{QC}}$ (JV3)	QC (viii)			
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)	$\boxed{\text{PH}}$ (Thm 3)	P (xiii)	$\boxed{\#\text{P}}$ (Thm 4)	
		IN (PROD)	P (xv)	PH (xvi)	$\boxed{\text{P}}$ (Thm 5)		
	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)	PH (xix)			
		IN (PROD)	QC (xxiii)	QC (xxiv)			

#P means that the problem of classically simulating the circuits is a #P-hard problem.

Classification of classical simulation complexities of extended Clifford circuits

			Weak		Strong		
			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
OUT (BITS)	NON-ADAPT	IN (PROD)	P (v)	$\boxed{\text{PH}}$ (JV7)	P (JV1)	$\boxed{\#\text{P}}$ (Thm 1)	#P (JV6)
OUT (BITS)	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)	$\boxed{\#\text{P}}$ (JV2)	$\boxed{\#\text{P}}$ (Thm 2)	#P (vii)
OUT (BITS)	ADAPT	IN (PROD)	$\boxed{\text{QC}}$ (JV3)	QC (viii)	#P (ix)	#P (x)	#P (xi)
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)	$\boxed{\text{PH}}$ (Thm 3)	P (xiii)	$\boxed{\#\text{P}}$ (Thm 4)	#P (xiv)
OUT (PROD)	NON-ADAPT	IN (PROD)	P (xv)	PH (xvi)	$\boxed{\text{P}}$ (Thm 5)	#P (xvii)	#P (xviii)
OUT (PROD)	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)	PH (xix)	#P (xx)	#P (xxi)	#P (xxii)
OUT (PROD)	ADAPT	IN (PROD)	QC (xxiii)	QC (xxiv)	#P (xxv)	#P (xxvi)	#P (xxvii)

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			WEAK(1)	WEAK(n)	STR(1)	STR(n)	STR
OUT (BITS)	NON-ADAPT	IN (BITS)	P (i)	P (ii)	P (iii)	P (iv)	$\boxed{\text{P}}$ (JV4)
		IN (PROD)	P (v)	$\boxed{\text{PH}}$ (JV7)	P (JV1)	$\boxed{\#\text{P}}$ (Thm 1)	#P (JV6)
	ADAPT	IN (BITS)	P (vi)	$\boxed{\text{P}}$ (JV5)	$\boxed{\#\text{P}}$ (JV2)	$\boxed{\#\text{P}}$ (Thm 2)	#P (vii)
		IN (PROD)	$\boxed{\text{QC}}$ (JV3)	QC (viii)	#P (ix)	#P (x)	#P (xi)
OUT (PROD)	NON-ADAPT	IN (BITS)	P (xii)	$\boxed{\text{PH}}$ (Thm 3)	P (xiii)	$\boxed{\#\text{P}}$ (Thm 4)	#P (xiv)
		IN (PROD)	P (xv)	PH (xvi)	$\boxed{\text{P}}$ (Thm 5)	#P (xvii)	#P (xviii)
	ADAPT	IN (BITS)	$\boxed{\text{P}}$ (Thm 6)	PH (xix)	#P (xx)	#P (xxi)	#P (xxii)
		IN (PROD)	QC (xxiii)	QC (xxiv)	#P (xxv)	#P (xxvi)	#P (xxvii)

Concluding remarks

- Whether we can classically simulate extended Clifford circuits efficiently depends delicately on the ingredients of the circuit
- Seemingly ‘modest’ changes to the ingredients can lead to large complexity changes
- Several extensions can be proven to be hard to simulate, under plausible complexity assumptions.
- Future work: any further extensions? How about other notions of simulation, like approximate simulation?

References

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