Quantum versus Classical Measures of Complexity on Classical Information

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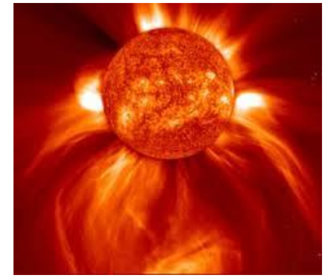
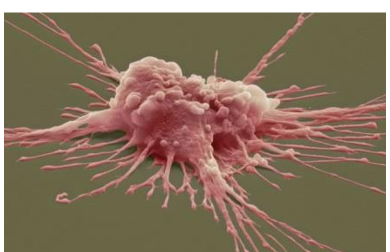
Astronomical Scale



Macroscopic Scale



Mesoscopic Scale

What is Complexity?

- Complexity is related to "Pattern" and "Organization".

> *Nature inherently organizes;*
> *Pattern is the fabric of Life.*
>
> James P. Crutchfield

- There are two extreme forms of Pattern: generated by a Clock and a Coin Flip.

- The former encapsulates the notion of Determinism, while the latter Randomness.

- Complexity is said to lie between these extremes.

J. P. Crutchfield and K. Young, *Physical Review Letters* **63**, 105 (1989).

Can Complexity be Measured?

- To measure Complexity means to measure a system's structural organization. How can that be done?

- Conventional Measures:
  - Difficulty in Description (in bits) – Entropy; Kolmogorov-Chaitin Complexity; Fractal Dimension.
  - Difficulty in Creation (in time, energy etc.) – Computational Complexity; Logical Depth; Thermodynamic Depth.
  - Degree of organization – Mutual Information; Topological ϵ-machine; Sophistication.

S. Lloyd, *"Measures of Complexity: a Non-Exhaustive List"*.

Topological ε-Machine

1. *Optimal* Predictor of the System's Process
2. *Minimal* Representation – Ockham's Razor
3. It is *Unique*.
4. It gives rise to a new measure of Complexity known as *Statistical Complexity* to account for the degree of organization.
5. The Statistical Complexity has an essential kind of *Representational Independence*.

J. P. Crutchfield, *Nature* **8**, 17 (2012).

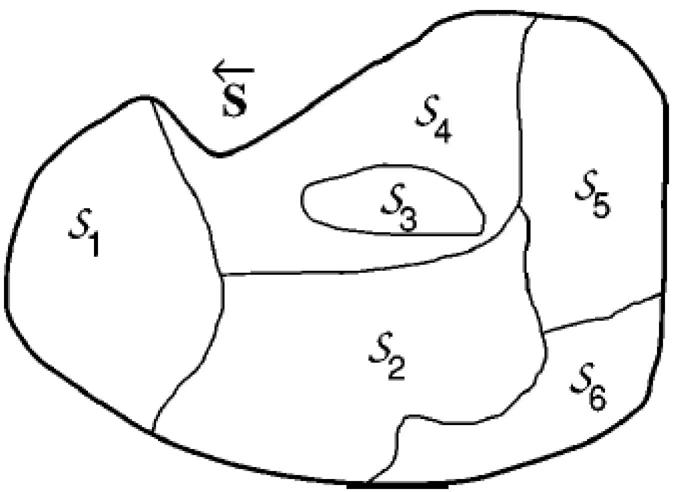
Concept of ε-Machine

Symbolic Sequences: ….. $S_{-2}S_{-1}S_0S_1$ …..

Futures : $\vec{S}_t = S_t S_{t+1} S_{t+2} \cdots$

Past : $\overleftarrow{S}_t = \cdots S_{t-3} S_{t-2} S_{t-1}$

Partitioning the set $\overleftarrow{\boldsymbol{S}}$ of all histories into causal states S_i.



- Within each partition, we have

$$\boxed{P\left(\vec{S} \mid \overleftarrow{s}\right) = P\left(\vec{S} \mid \overleftarrow{s}'\right)} \qquad (1)$$

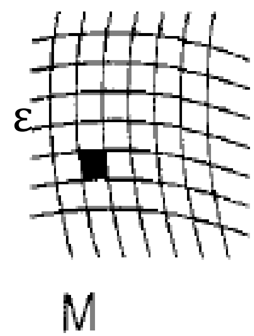
where $\overleftarrow{s}$ and $\overleftarrow{s'}$ are two different individual histories in the same partition.

- Equation (1) is related to sub-tree similarity discussed later.

J. P. Crutchfield and C. R. Shalizi, *Physical Review E* **59**, 275 (1999).

ε Machine - Preliminary

- In physical processes, measurements are taken as time evolution of the system.
- The time series is converted into a sequence of symbols $s = s_1, s_2, \ldots, s_i, \ldots$ at time interval τ.



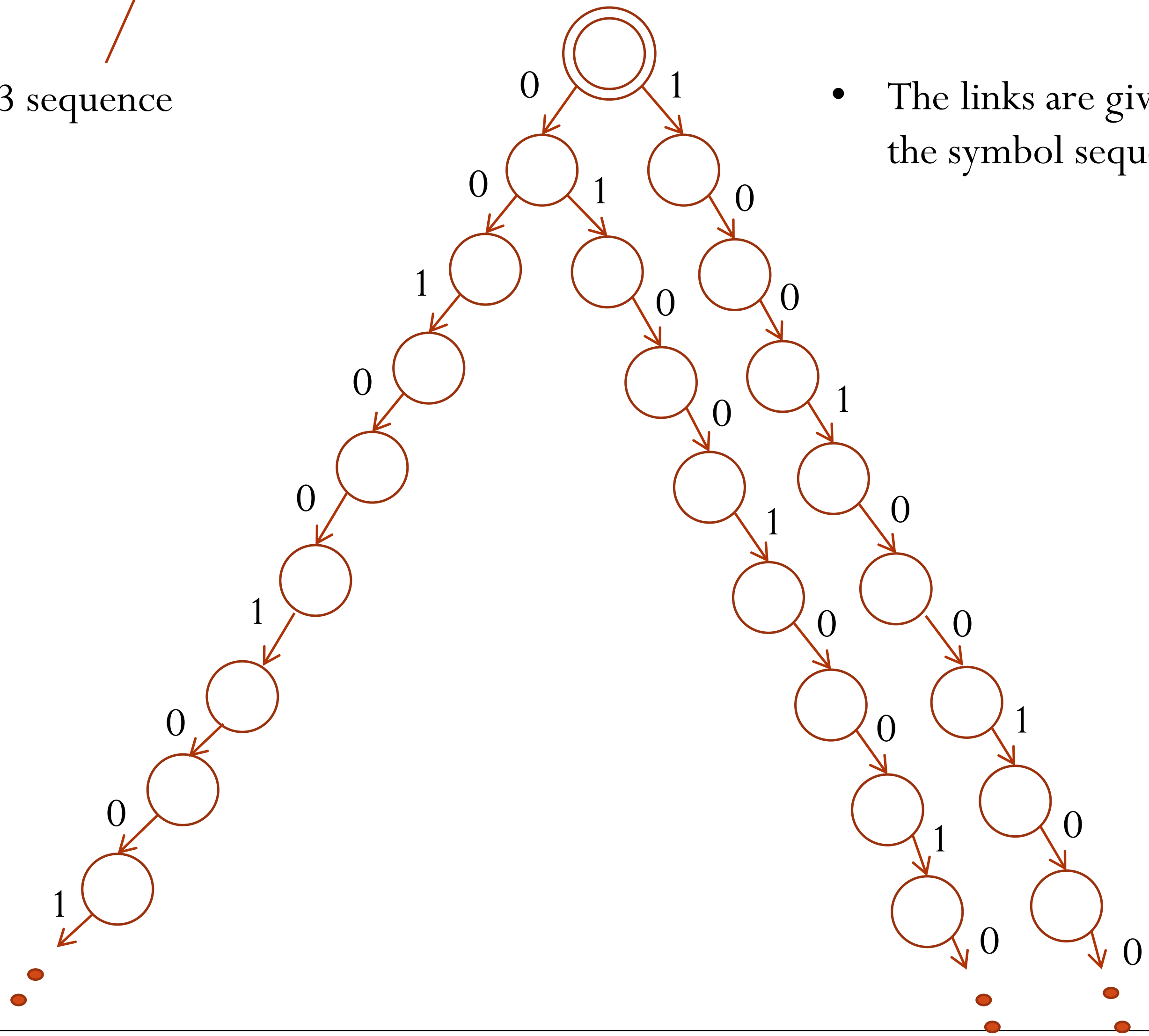
- Measurement phase space M is segmented or partitioned into cells of size ε.
- Each cell can be assigned a label leading to a list of alphabets $A = \{0, 1, 2, \ldots k - 1\}$.
- Then, each $s_i \in A$.

Ɛ-Machine – Reconstruction

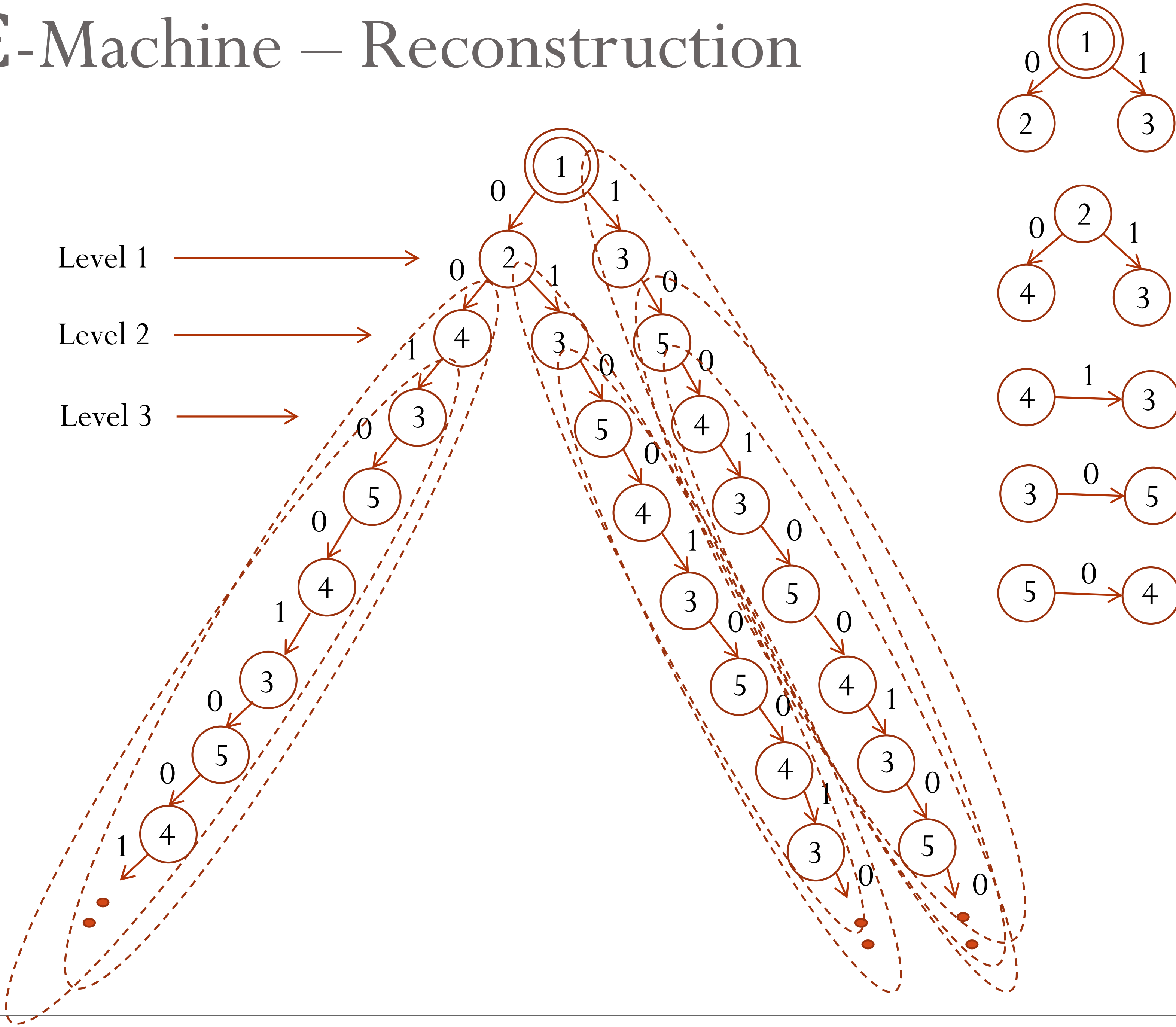
S = 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, …..

Period-3 sequence

- From symbol sequence -> build a *tree structure* with nodes and links .
- The links are given symbols according to the symbol sequence with $A = \{0, 1\}$.



ε-Machine – Reconstruction

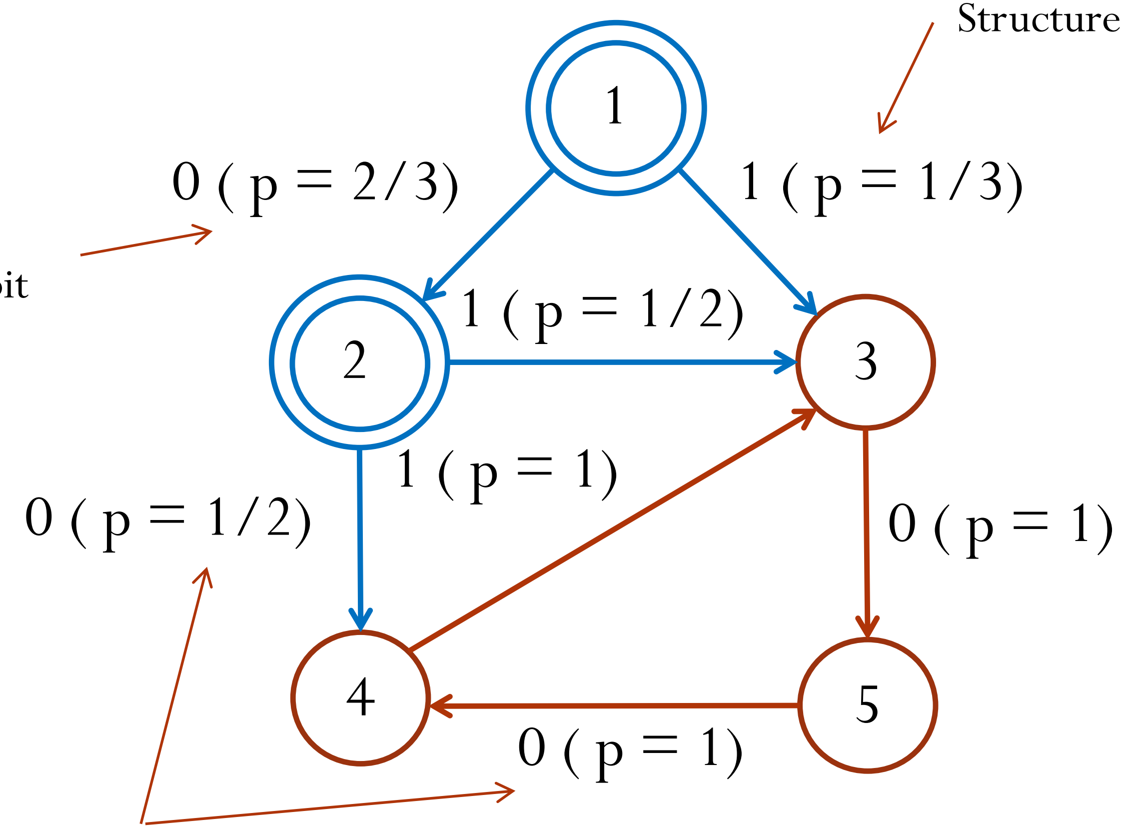


ε-Machine – Reconstruction

S = 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, …..

Probabilistic Structure

Based on Orbit Ensemble



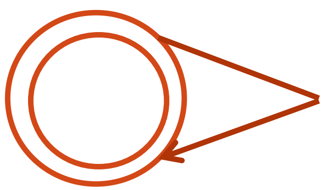
Combination of both deterministic and random computational resources

But the Probabilistic Structure is transient.

ε-Machine – Simplest Cases

All Head or All “1” Process

s = 1, 1, 1, 1, 1, 1, …..

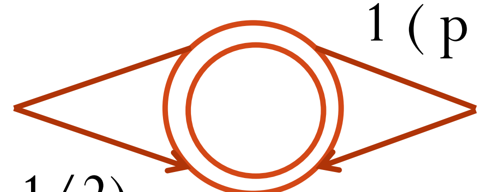


1 (p = 1)

Purely Random Process

Fair Coin Process

1 (p = 1/2)

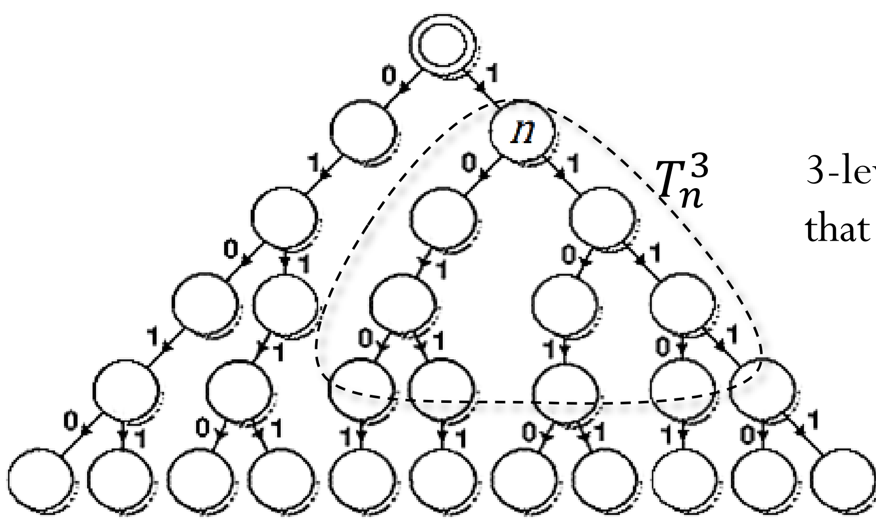


0 (p = 1/2)

- These are the extreme cases of Complexity.
- They are structurally simple.
- We expect their Complexity Measure to be zero.

ε-Machine:
Complexity from Deterministic Dynamics

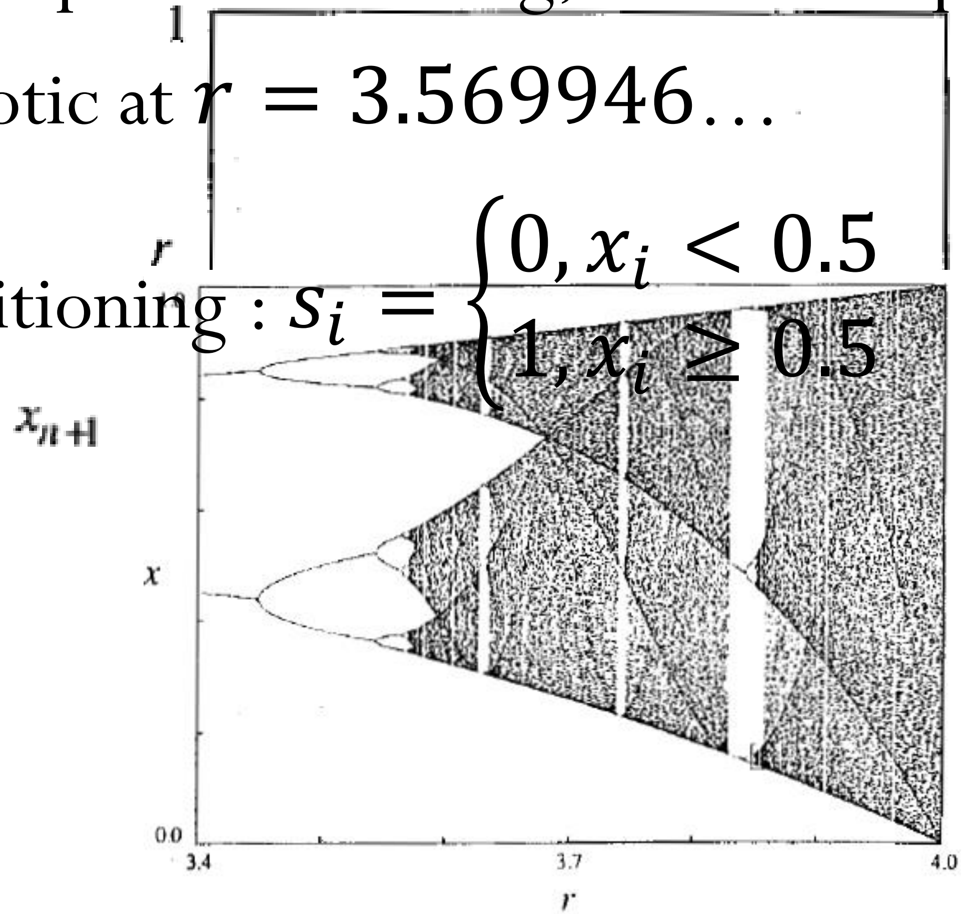
- The Symbol Sequence is derived from the Logistic Map.
- The parameter *r* is set to the band merging regime $r = 3.67859\ldots$
- For a finite sequence, a tree length *Tlen* and a machine length *Mlen* need to be defined.



3-level sub-tree of *n* contains all nodes that can be reached within 3 links

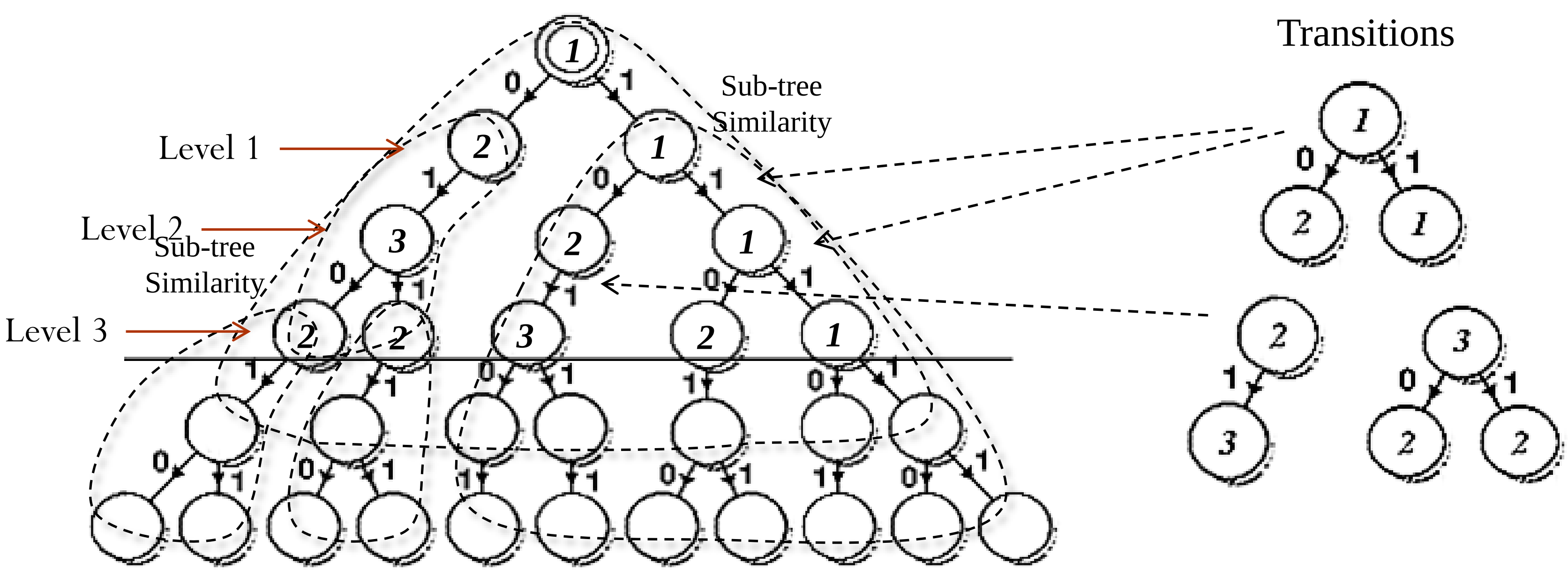
The Logistic Map

- Logistic equation : $x_{n+1} = r x_n (1 - x_n)$
- Starts period-doubling; at $r = 3$: period-2
- Chaotic at $r = 3.569946\ldots$
- Partitioning : $s_i = \begin{cases} 0, x_i < 0.5 \\ 1, x_i \geq 0.5 \end{cases}$

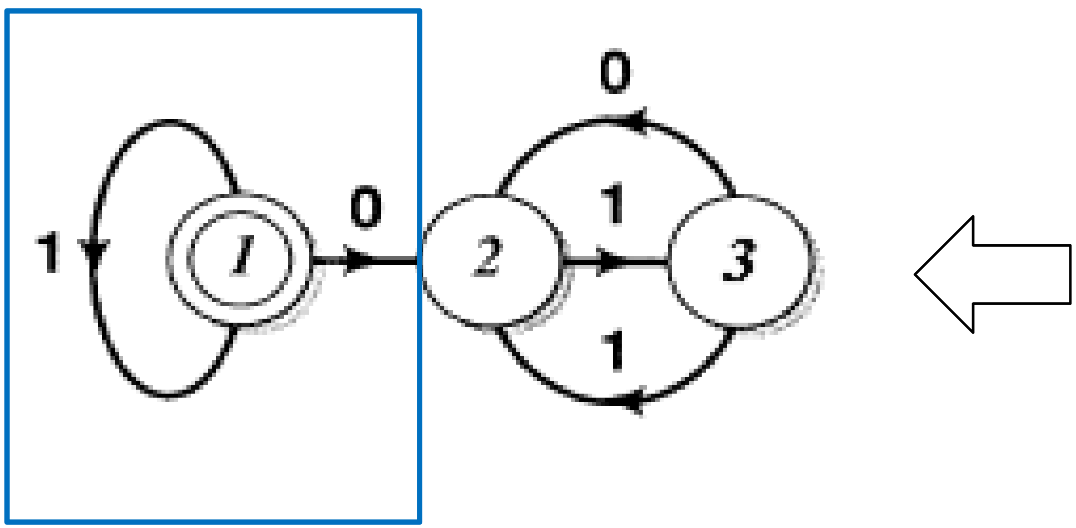


ε-Machine:
Complexity from Deterministic Dynamics

Construction of ε-machine:

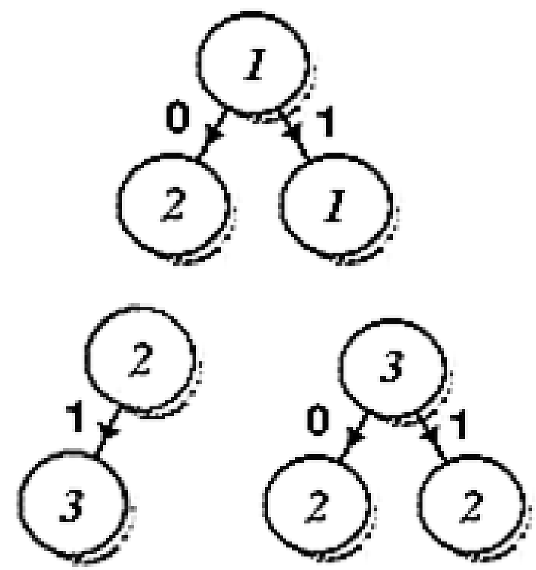


Ɛ-Machine:
Complexity from Deterministic Dynamics

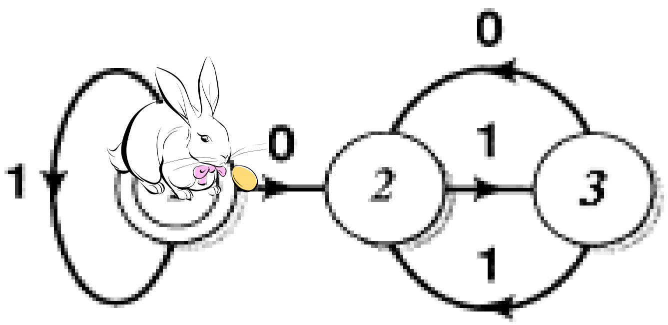


Transient State

Transitions



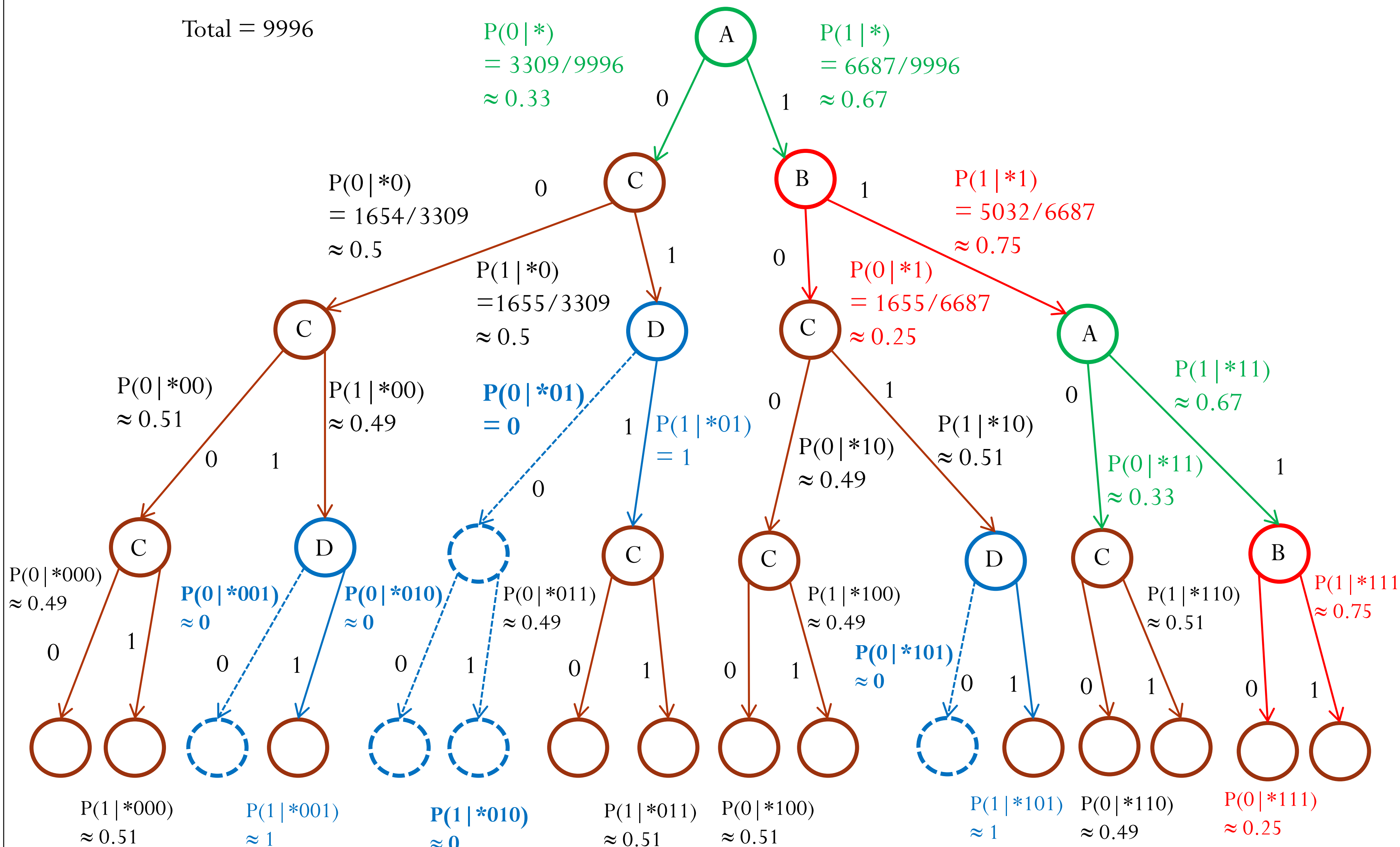
ε-Machine:
Complexity from Deterministic Dynamics



$s =$

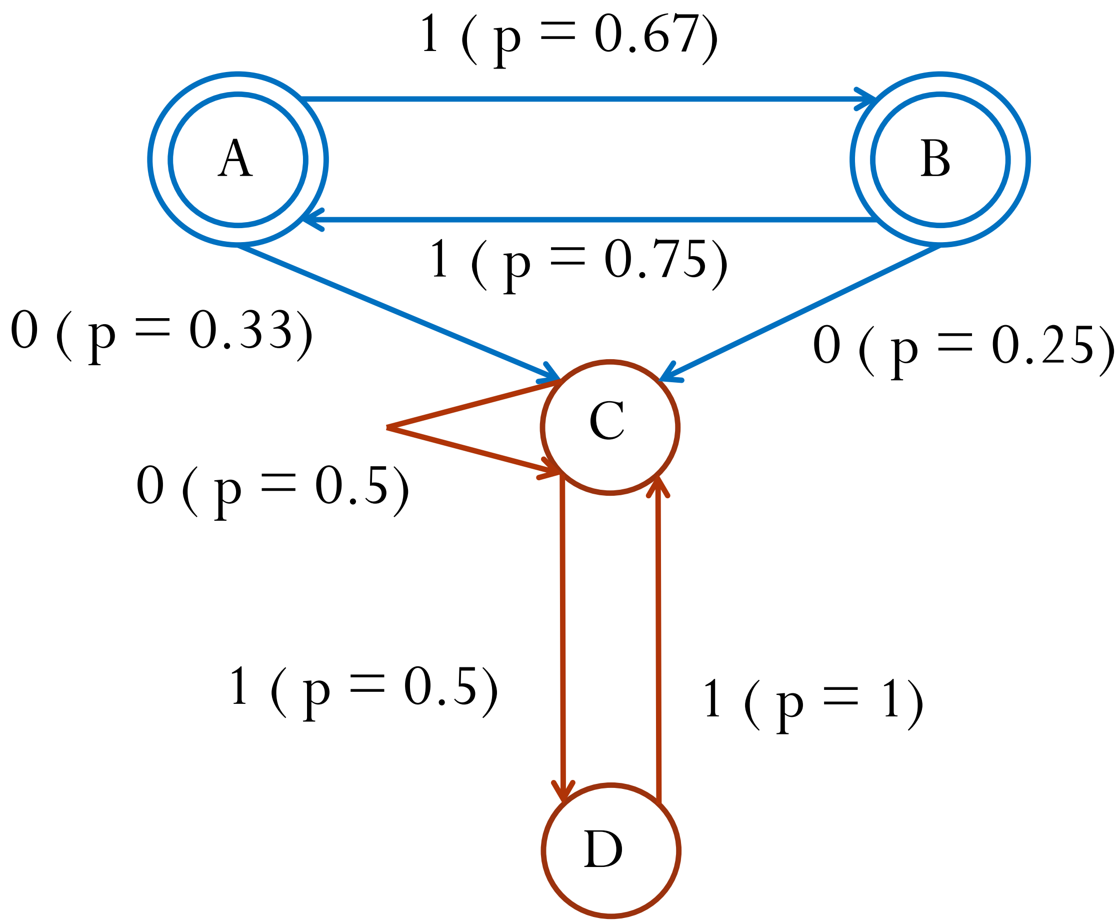
- The ε-machine captures the patterns of the process.

ε-Machine: Causal State Splitting Reconstruction



C. R. Shalizi, K. L. Shalizi and J. P. Crutchfield, *arXiv preprint cs/0210025*.

ε-Machine: Causal State Splitting Reconstruction - The Even Process



Statistical Complexity and Metric Entropy

- Statistical Complexity is defined as

$$\boxed{C_\mu = -\sum_{\{\sigma\}} p(\sigma)\log_2 p(\sigma) \quad \text{Bits}}$$

It serves to measure the computational resources required to reproduce a data stream.

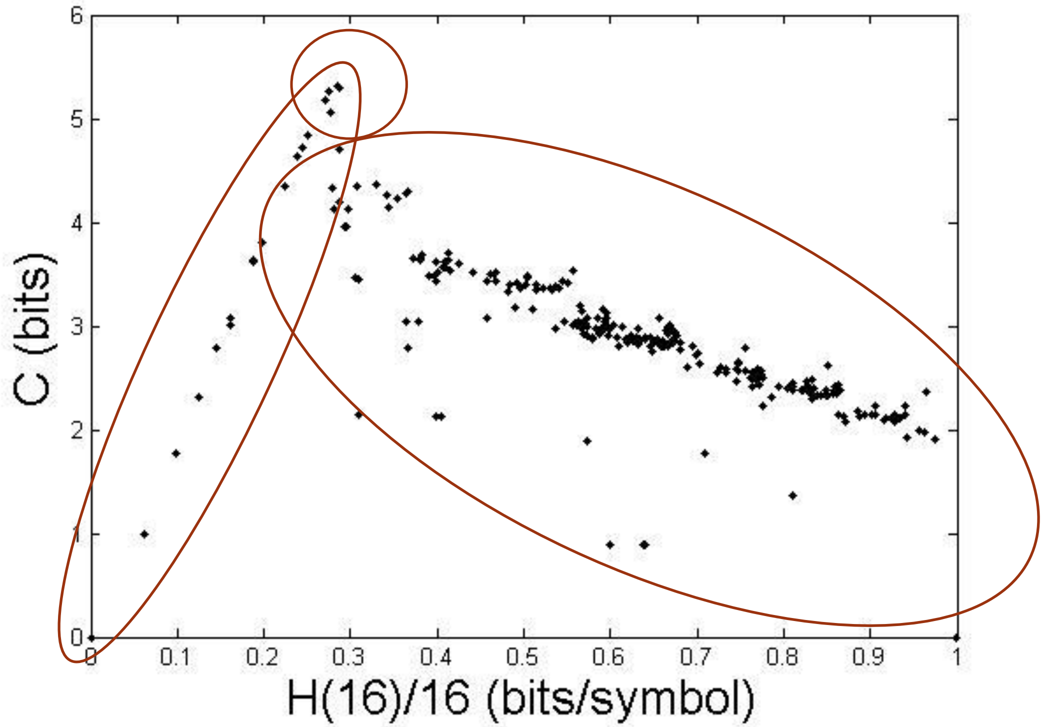
- Metric Entropy is defined as

$$h = -\frac{1}{m}\sum_{\{s^m\}} p(s^m)\log_2 p(s^m) \qquad \text{Bits/Symbols}$$

It serves to measure the diversity of observed patterns in a data stream.

J. P. Crutchfield, *Nature Physics* **8**, 17 (2012).

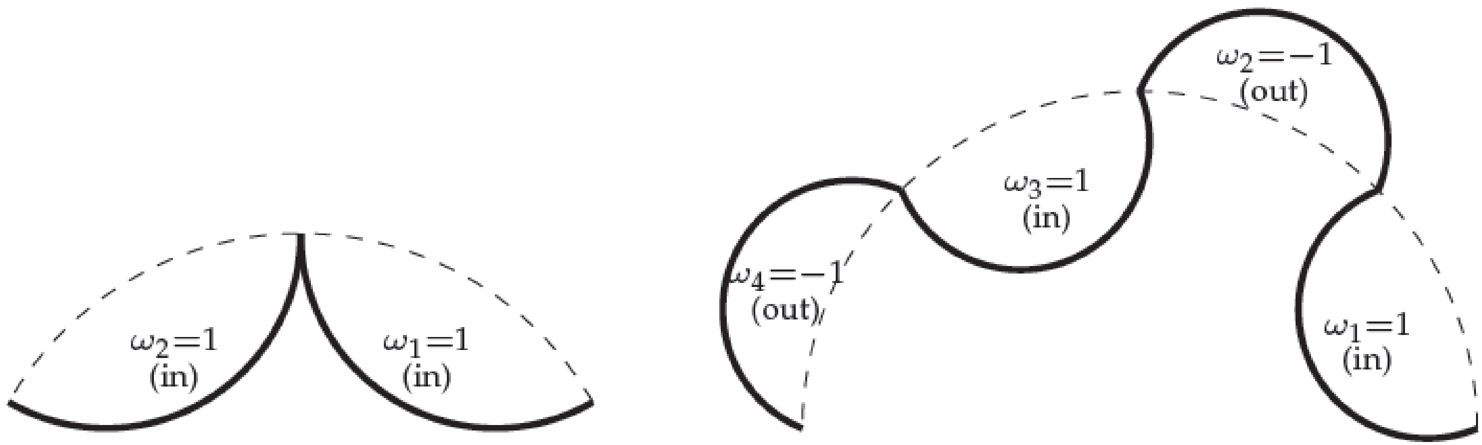
The Logistic Map



- Linear region in the periodic window
- Decreasing trend at chaotic region
- Phase transition, $H^* \approx 0.28$: "edge of chaos"

The Arc-Fractal Systems

A novel idea to generate fractals with "tunable" dimension[1]



(a) $\alpha = \frac{2\pi}{3}, n = 2$ and $\boldsymbol{\omega} = (\omega_1, \omega_2) = (1, 1)$

(b) $\alpha = \pi, n = 4$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4) = (1, -1, 1, -1)$

Figure : Basic idea of the arc-fractal system.

- The parameters: α (angle), n (number of arcs) and $\boldsymbol{\omega}$ (orientation of arcs)
- Three types of rule: single, multiple and random

[1] H. N. Huynh and L. Y. Chew, *Fractals*, **19** 141 (2011).

The Arc-Fractal Systems

Different parameters yield different fractals[1]

Table : The combinations of α, n, and $\boldsymbol{\omega}$ that give the classic fractals.

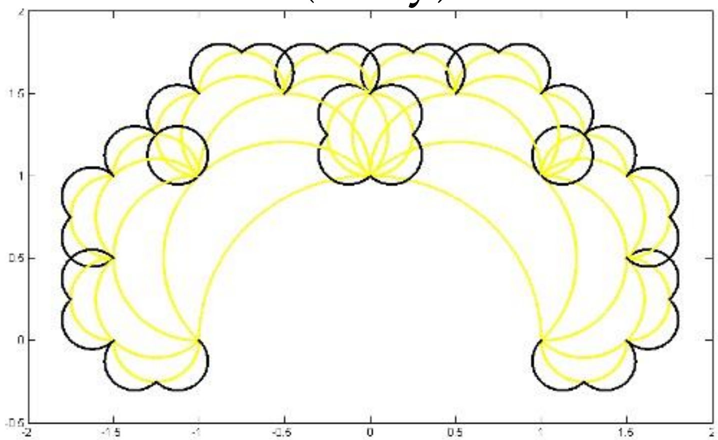
Classic fractals	α	n	$\boldsymbol{\omega}$
Koch	$\frac{2\pi}{3}$	2	$(+1;+1)$
Heighway	π	2	$(-1;+1)$
Lévy	π	2	$(-1;-1)$
Sierpiński	π	3	$(+1;-1;+1)$
Eisenstein*	π	3	$(-1;+1;-1)$

*Eisenstein fraction is created by joining three pieces of crab fractals together.

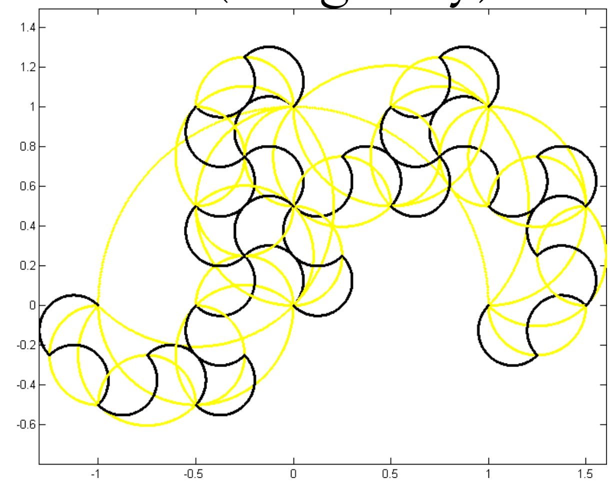
[1] H. N. Huynh and L. Y. Chew, *Fractals*, **19** 141 (2011).

The Arc-Fractal Systems

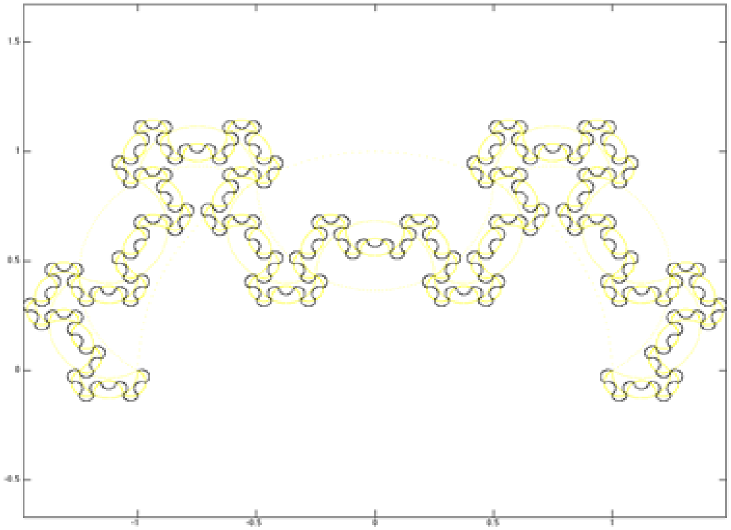
- Out-Out (Levy):



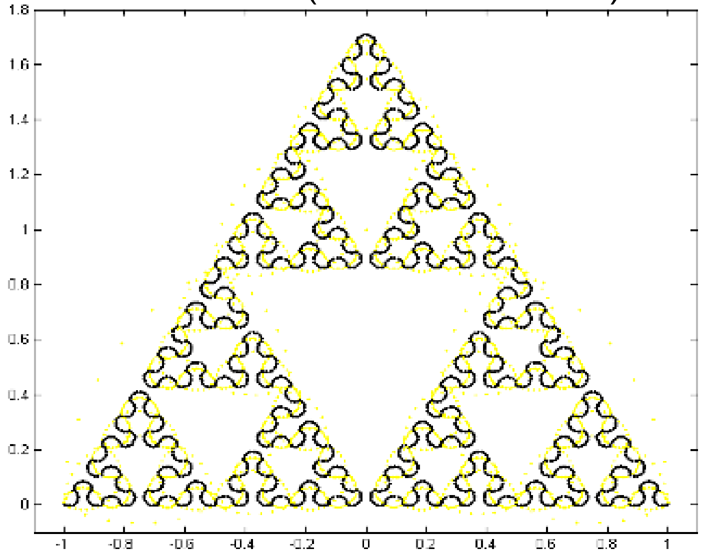
- Out-In (Heighway):



- Out-In-Out (Crab):

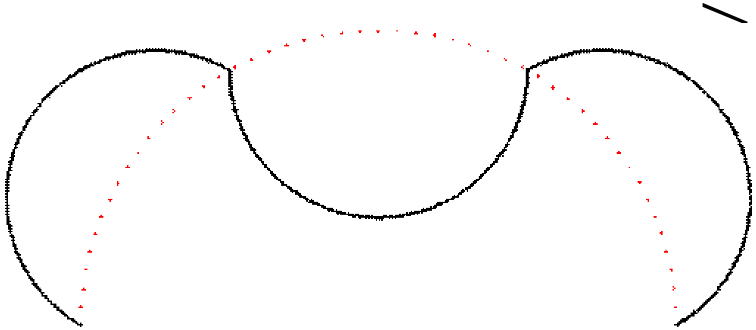


- In-Out-In (Arrowhead):



The Arc-Fractal Systems

- Sequence -> each arc is given an index according to its angle of orientation



Sequence : 11,7,3

Out-In-Out rule, level 3 of construction:

(7,3,11,7,11,3,11,7,3,11,3,7,11,7,3,7,11,3,11,7,3,11,3,7,3,11,7)

In base-3:

(1,0,2,1,2,0,2,1,0,2,0,1,2,1,0,1,2,0,2,1,0,2,0,1,0,2,1)

The Arc-Fractal Systems

- The fractals generated by the arc-fractal systems can be associated with symbolic sequences[1,2].

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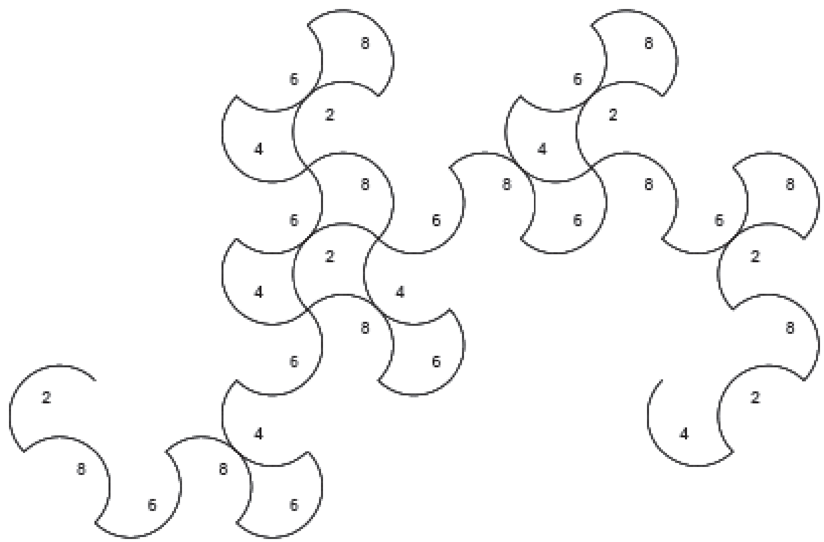


[1]H. N. Huynh and L.Y. Chew, *Fractals* **19**, 141 (2011).

[2]H. N. Huynh, A. Pradana and L.Y. Chew, *PLoS ONE* **10**(2), e0117365 (2015)

The Arc-Fractal Systems

Table : Results of arc-fractal sequences with 2 arc segments (Heighway and Lévy) and 3 arc segments (arrowhead and crab).[1]

Fractal	d	n	ω	number of symbols†	h (bit/symbol)	C_S (bit)
Lévy	2	2	$(-1;-1)$	4	0.389	9.80
Heighway	2	2	$(-1;1)$	4	0.359	9.27
Arrowhead	$\frac{\ln 3}{\ln 2}$	3	$(1;-1;1)$	3	0.387	7.34
Crab	$\frac{\ln 3}{\ln 2}$	3	$(-1;1;-1)$	3	0.387	7.34

H. N. Huynh, A. Pradana and L.Y. Chew, *PLoS ONE* **10**(2), e0117365 (2015)

Complex Symbolic Sequence and Neuro-behavior

The complex symbolic sequences can be employed as stimuli in neurobehavioural or human pattern recognisation experiments.[1]

- Map each symbol to a corresponding physical state
- Whether and how the human brain can perceive or predict the states

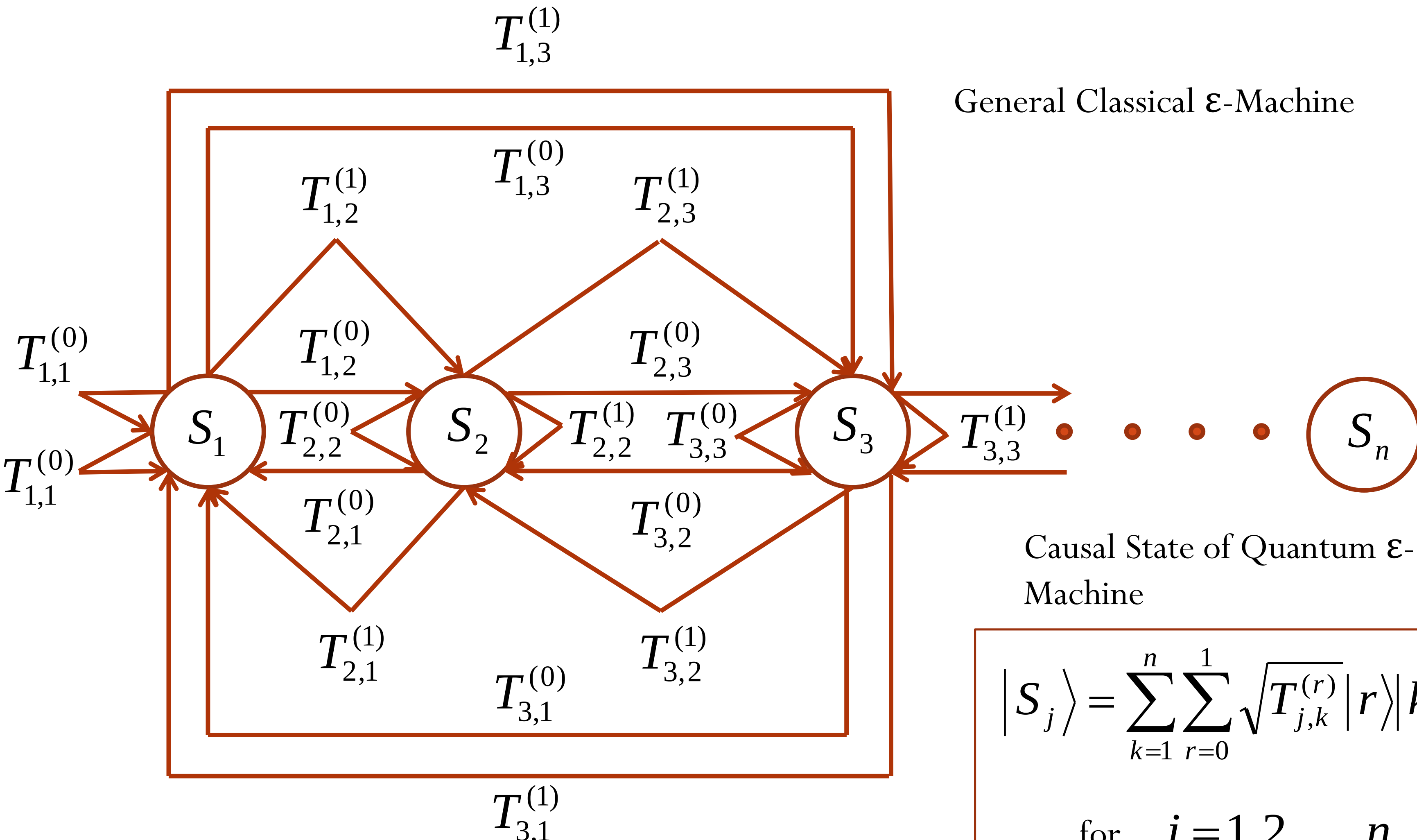
What is the next symbol in the sequence

$$\diamondsuit\clubsuit\spadesuit\clubsuit\spadesuit\heartsuit\spadesuit\clubsuit\spadesuit\heartsuit\diamondsuit\heartsuit\spadesuit\heartsuit\spadesuit\clubsuit\spadesuit\heartsuit\diamondsuit\heartsuit\diamondsuit \ldots$$

a. $\clubsuit$ b. $\diamondsuit$ c. $\heartsuit$ d. $\spadesuit$? How confident are you?

Difficulty quantified by measure of complexity?

Quantum ε-Machine



General Classical ε-Machine

Causal State of Quantum ε-Machine

$$\left|S_j\right\rangle = \sum_{k=1}^{n}\sum_{r=0}^{1}\sqrt{T_{j,k}^{(r)}}\left|r\right\rangle\left|k\right\rangle$$

for $j = 1, 2, \ldots, n$

M. Gu, K. Wiesner, E. Rieper and V. Vedral, *Nature Communications* **3**, 762 (2012)

Quantum Complexity

Quantum Causal State :

$$\left|S_j\right\rangle = \sum_{k=1}^{n}\sum_{r=0}^{1}\sqrt{T_{j,k}^{(r)}}\left|r\right\rangle\left|k\right\rangle \quad \text{for} \quad j = 1, 2, \ldots, n$$

Quantum Complexity :

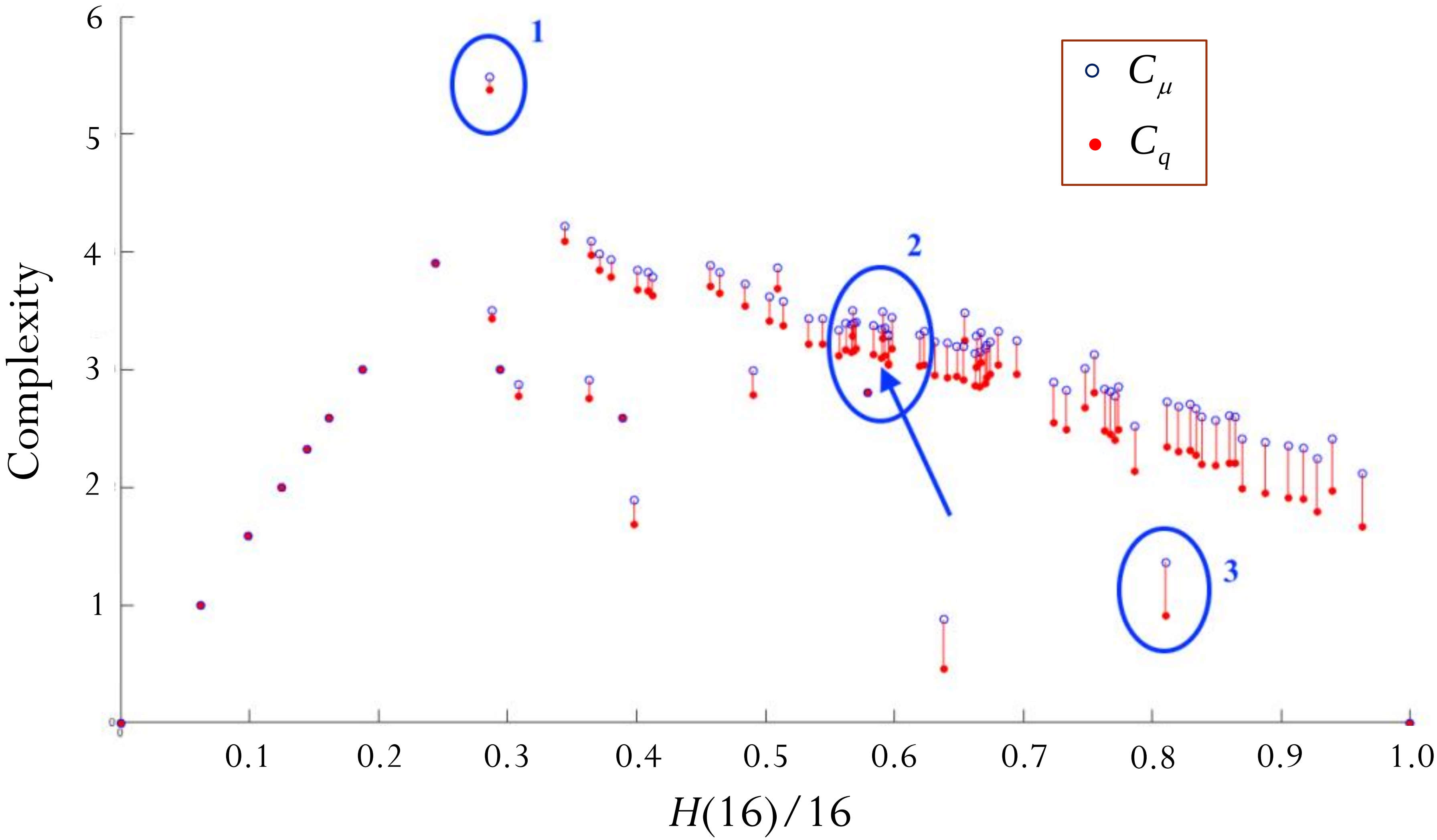
$$\boxed{C_q = -\mathrm{Tr}\left(\rho \log \rho\right) \ \text{Bits}}$$

where the density matrix $\rho = \sum_i p_i \left|S_i\right\rangle\left\langle S_i\right|$ and p_i is the probability of the quantum causal state $\left|S_i\right\rangle$.

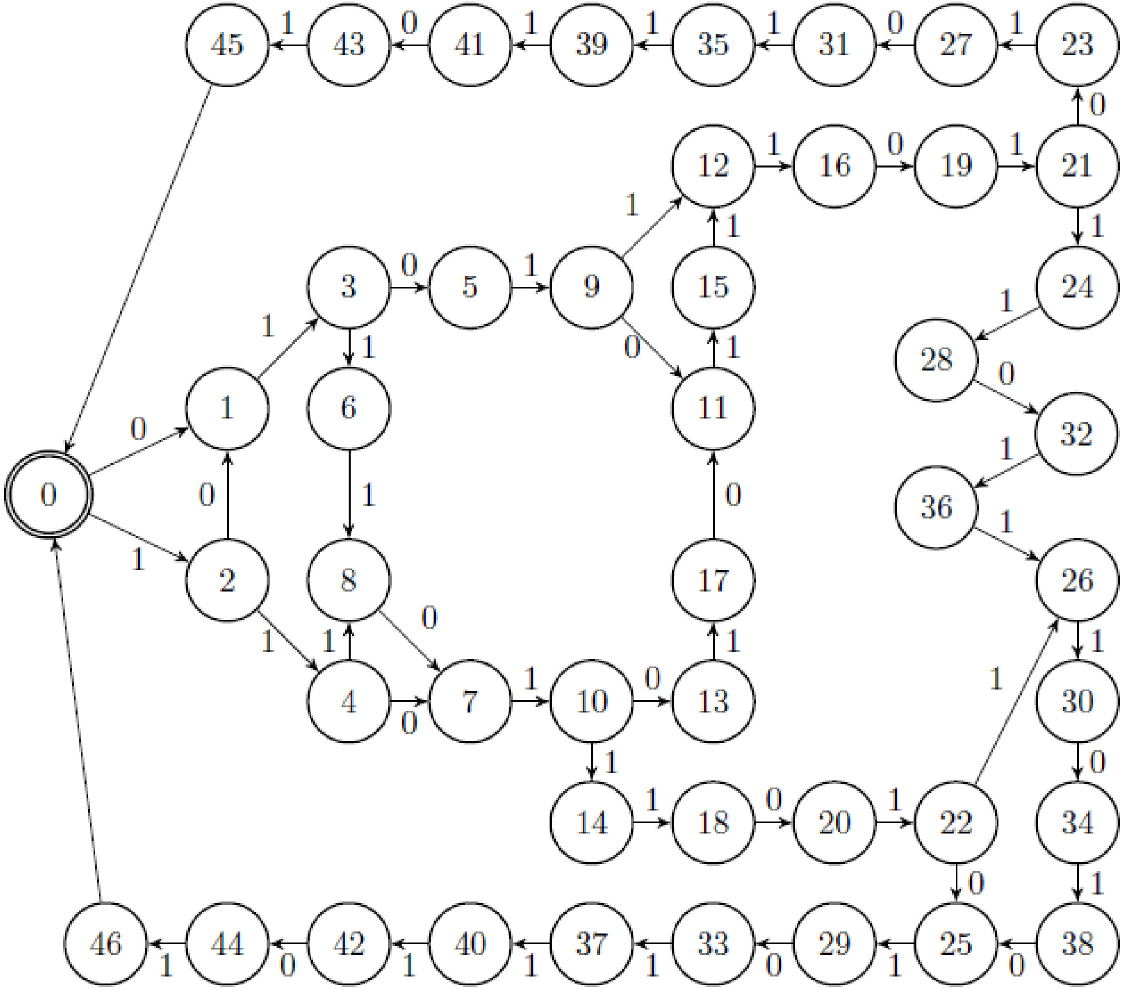
Note that $C_q \leq C_\mu$ where $C_\mu = -\sum_{\{\sigma\}} p(\sigma)\log_2 p(\sigma)$ Bits

R. Tan, D. R. Terno, J. Thompson, V. Vedral and M. Gu, *European Physical Journal Plus* **129**(9), 1-10 (2014)

Quantum versus Classical Complexity
The Logistic Map



Ɛ-Machine at Point 1 – Edge of Chaos

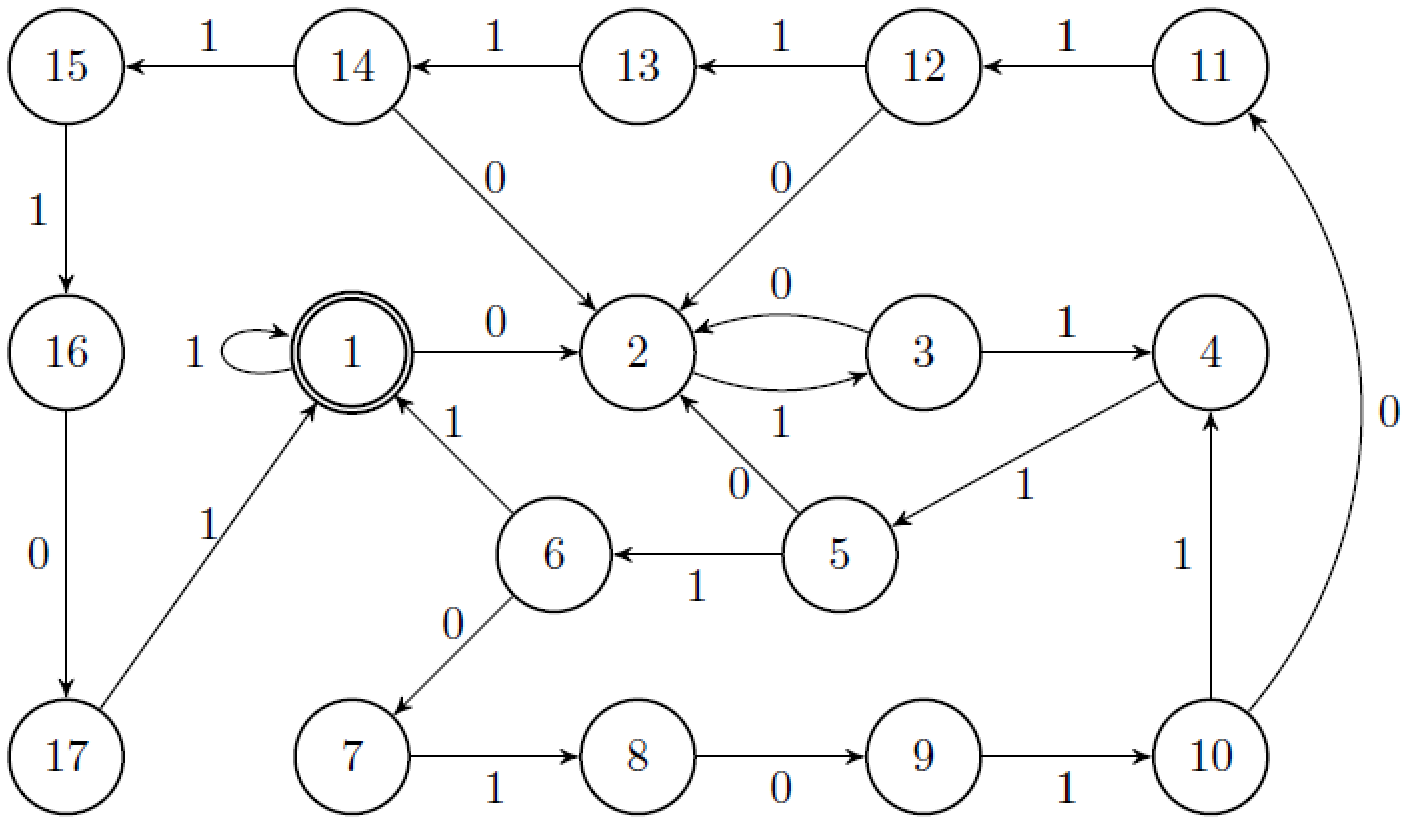


$R = 3.569946....$

$C_\mu = 5.480236$

$C_q = 5.376835$

Ɛ-Machine at Point 2 – Chaotic Region



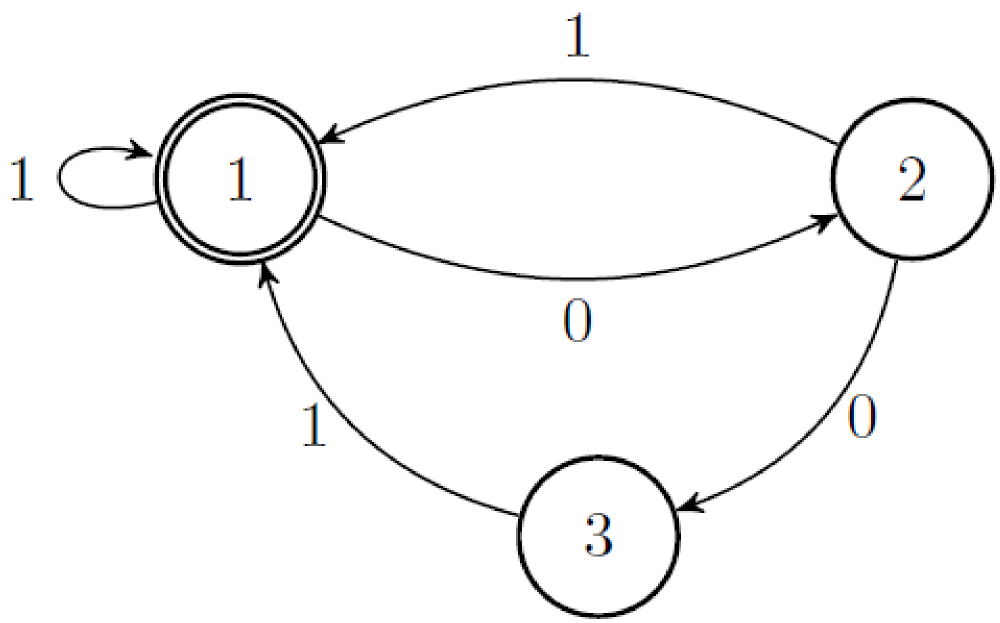
$R = 3.715$

$C_\mu = 3.341681$

$C_q = 3.093695$

Ɛ-Machine at Point 3 – Chaotic Region

$$R = 3.960$$



$$\boxed{C_\mu = 1.364846}$$

$$\boxed{C_q = 0.906989}$$

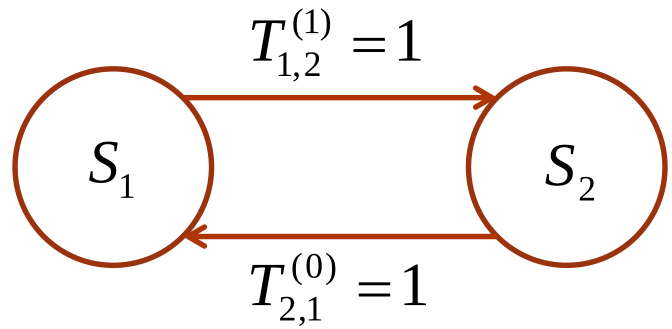
Quantum Ɛ-Machine -Change of Measurement Basis

General Qubit Measurement Basis:

$$|\alpha\rangle = a|0\rangle + b|1\rangle$$
$$|\beta\rangle = b^*|0\rangle - a^*|1\rangle$$

where $|\alpha\rangle$ and $|\beta\rangle$ are new orthogonal measurement basis such that:

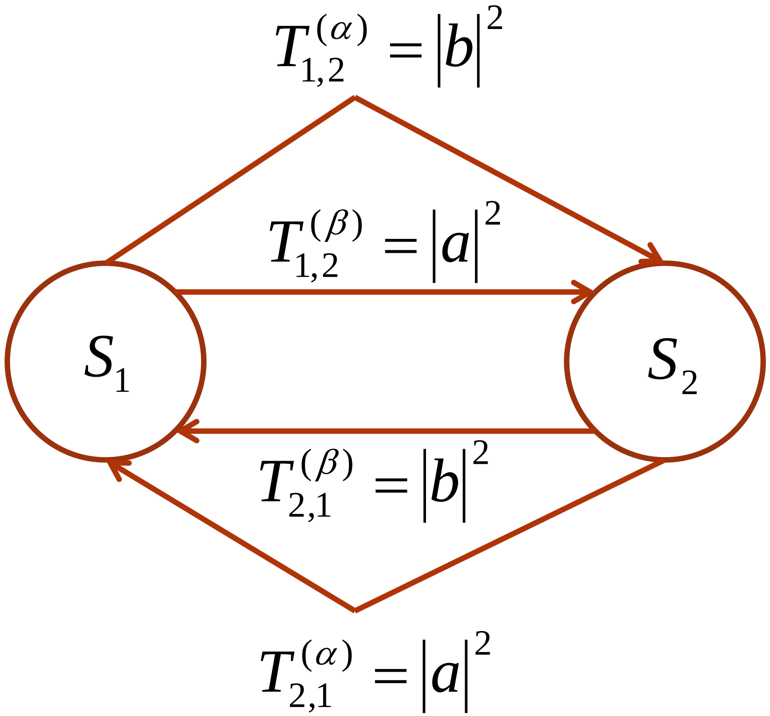
$$\boxed{\begin{aligned} |0\rangle &= a^*|\alpha\rangle + b|\beta\rangle \\ |1\rangle &= b^*|\alpha\rangle - a|\beta\rangle \end{aligned}}$$



$$|S_1\rangle = |1\rangle|2\rangle$$
$$|S_2\rangle = |0\rangle|1\rangle$$

Quantum Ɛ-Machine
-Change of Measurement Basis

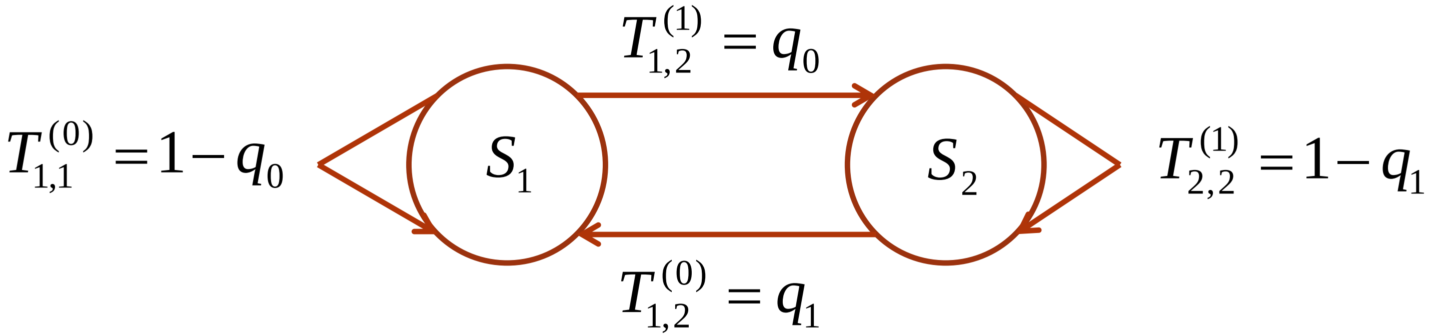
$$|S_1\rangle = b^*|\alpha\rangle|2\rangle - a|\beta\rangle|2\rangle$$
$$|S_2\rangle = a^*|\alpha\rangle|1\rangle + b|\beta\rangle|1\rangle$$



Deterministic ε - Machine

Stochastic ε - Machine

Quantum Ɛ-Machine -Change of Measurement Basis



Stochastic ε - Machine

$$|S_1\rangle = (1-q_0)|0\rangle|1\rangle + q_0|1\rangle|2\rangle$$
$$|S_2\rangle = q_1|0\rangle|1\rangle + (1-q_1)|1\rangle|2\rangle$$

Quantum Ɛ-Machine
-Change of Measurement Basis

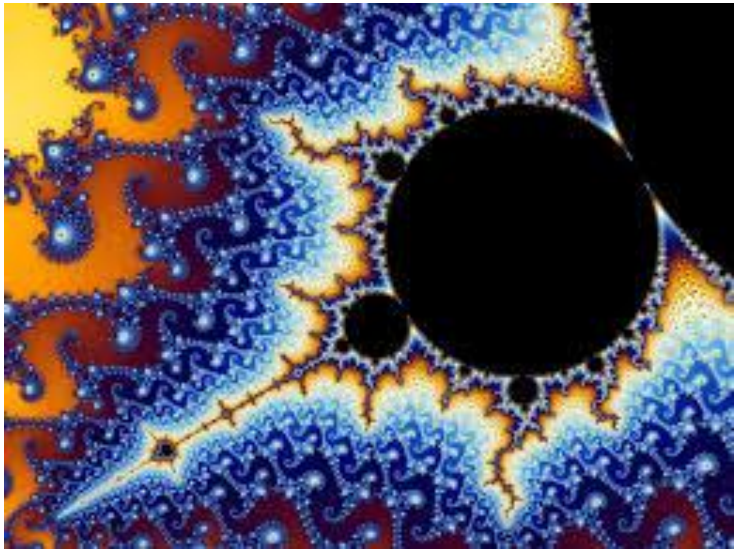
$$\left|S_1\right\rangle = \left\{(1-q_0)a^*\left|1\right\rangle + q_0\, b^*\left|2\right\rangle\right\}\left|\alpha\right\rangle + \left\{(1-q_0)b\left|1\right\rangle - q_0\, a\left|2\right\rangle\right\}\left|\beta\right\rangle$$
$$\left|S_2\right\rangle = \left\{q_1\, a^*\left|1\right\rangle + (1-q_1)b^*\left|2\right\rangle\right\}\left|\alpha\right\rangle + \left\{q_1\, b\left|1\right\rangle - (1-q_1)a\left|2\right\rangle\right\}\left|\beta\right\rangle$$

If the quantum ε – machine is prepared in the quantum causal state $\left|S_1\right\rangle$ and measurement yields , $\left|\alpha\right\rangle$ the quantum ε – machine now possesses the causal state:

$$(1-q_0)a^*\left|S_1\right\rangle + q_0\, b^*\left|S_2\right\rangle$$

which is a superposition of the quantum resource. Further measurements in the $\left|\alpha\right\rangle$ and $\left|\beta\right\rangle$ basis leads to further iterations within the above results due to the inherent entangling features in the quantum causal state structure.

Ideas to Proceed



Mandelbrot Set

- *Complexity within Simplicity*

Question:
Could there be infinite regress in terms of information processing within quantum ε – Machine?

Question:
Could classical ε – Machine acts as a mind-reading machine in the sense of Claude Shannon's? Would a quantum version do better?

Neil Huynh

Andri Pradana

Matthew Ho

Thank You