Quantum Precision:
Unbounded memory advantage in stochastic simulation using quantum mechanics.

Andrew J. P. Garner

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John Templeton Foundation

FQXi
Foundational Questions Institute

NUS National University of Singapore

CQT Centre for Quantum Technologies

Collaborators

Liu Qing

Mile Gu

Jayne Thompson

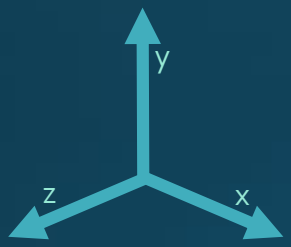
Andrew Garner

Vlatko Vedral

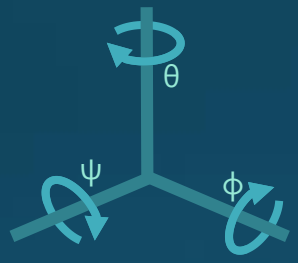
Motivation

Real numbers are ubiquitous

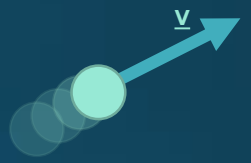
We often want to simulate systems whose properties are described by real numbers:



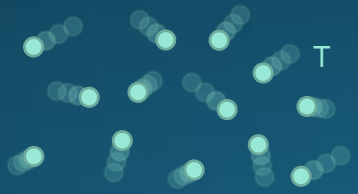
Positions



Angles



Velocities

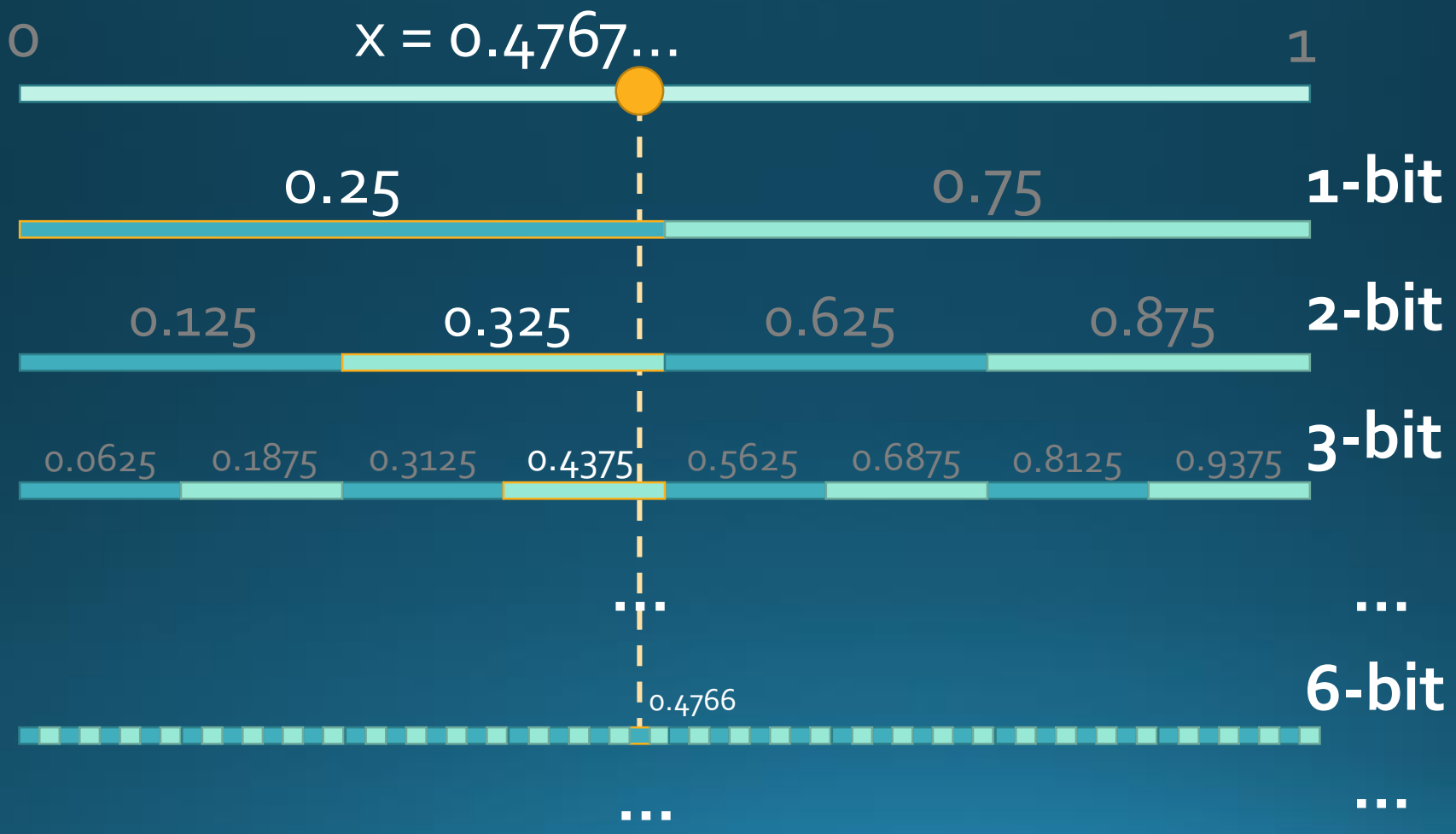


Temperatures

Field strengths

Digital computers cannot directly store real numbers

Precision is memory-limited



Precision and memory cost

There are two subtly distinct concepts:

Precision – the closeness of a number to some 'true' value – quantified by the number of divisions $N = 2^n$ (statistical property of simulation)

Memory cost – the amount of information required to store a value (physical property of simulator)

In classical information theory, these are so fundamentally related that they are used almost interchangeably (e.g. "32-bit precision")

In quantum mechanics these two quantities can diverge significantly!

Scenario

Scenario: cyclic random walks

Consider a bead on a ring:

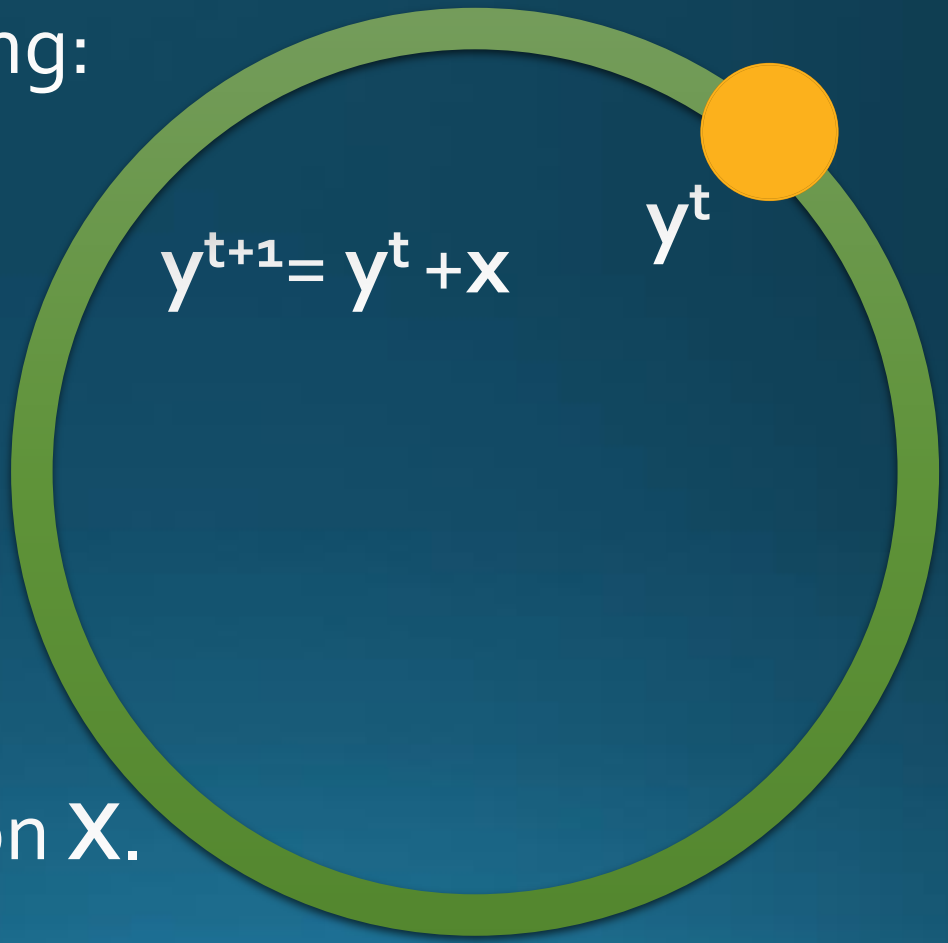
Position y^t at time **t**.

Discrete time-steps.

Position at time **t+1**:

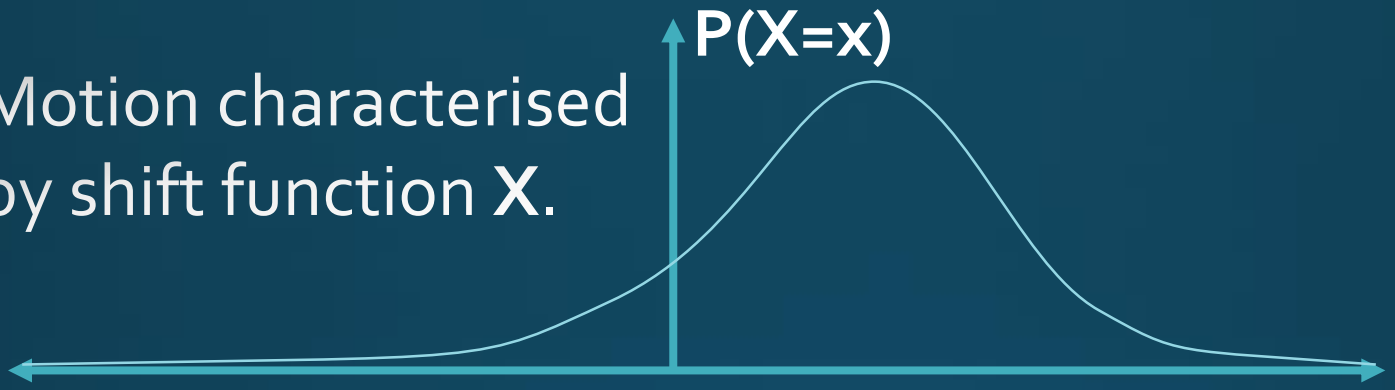
$$y^{t+1} = y^t + x$$

where **x** is randomly chosen from some (continuous) distribution **X**.

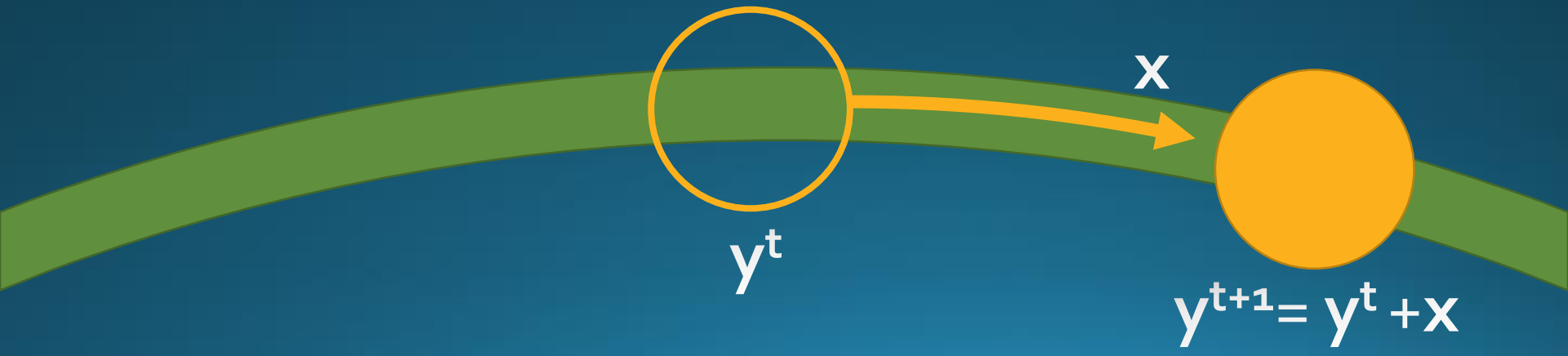


Scenario: cyclic random walks

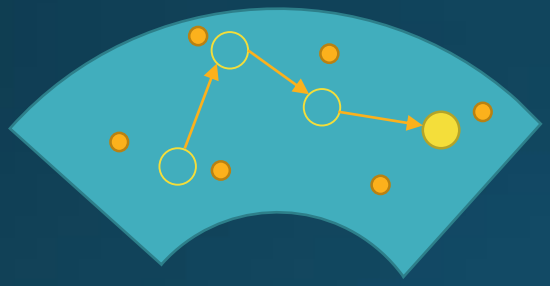
Motion characterised by shift function **X**.



Path described by series of continuous random variables $Y^0, Y^1 \ldots Y^t \ldots$ obeying $Y^{t+1} = Y^t + X \pmod 1$

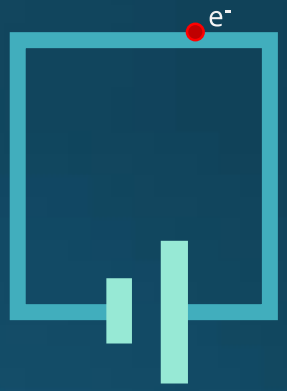


Scenario: cyclic random walks



Azimuthal motion of diffusing gas

Bead on ring



Single electron in circuit

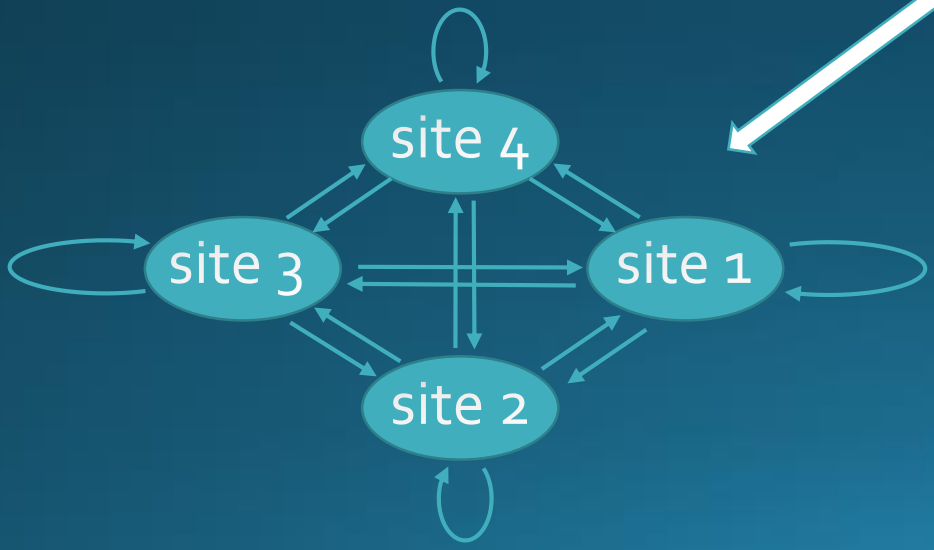
Any system with symmetry and diffusion

Classical simulators

To simulate to precision n, the ring is split into $N=2^n$ *sites*, {s}.

Shift function X approximated as N×N *stochastic matrix.*

Simulation is effectively a finite state machine:



Statistics of future positions Y^{t+1}, Y^{t+2}, … depend on the current site s^t.

This information must be stored in the simulator's memory!

The cost of classical simulation

More divisions gives greater statistical accuracy of simulation.

The state machine's memory must accommodate more possibilities.

For process with randomness, in steady state each site is equally likely.

Entropy cost of simulating to precision $n = \log_2 N$ is then:

$$H_{\text{classical}} = n \quad \text{(bits)}$$

Quantum simulator

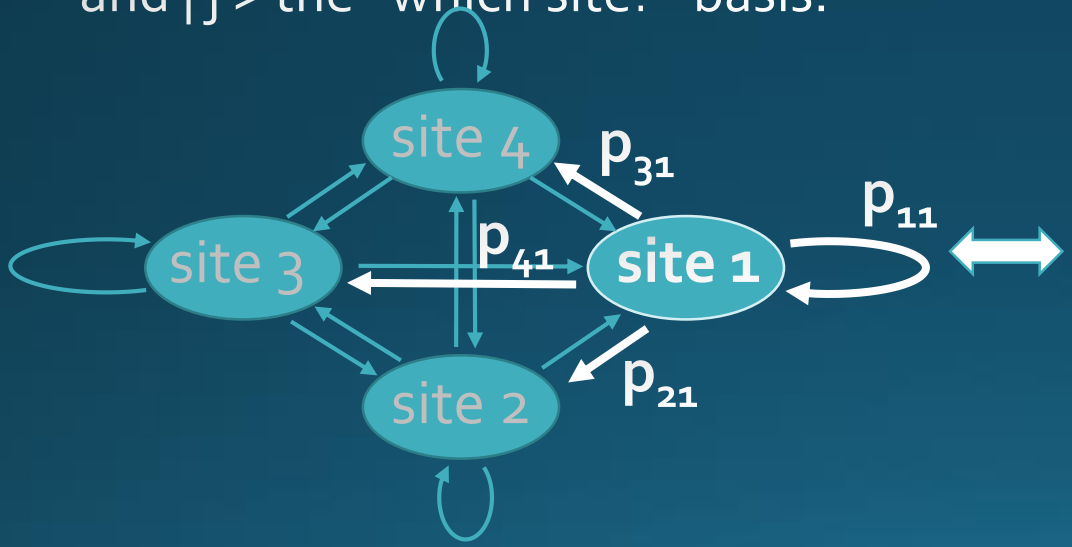
Quantum simulator

For a given precision n, each site is mapped onto a quantum state:

$$|S_i\rangle = \sum_{j=1}^{2^n} \sqrt{p_{ji}}\,|j\rangle$$

where p_{ji} is the probability of transitioning from site i to j,

and $|j\rangle$ the "which site?" basis.



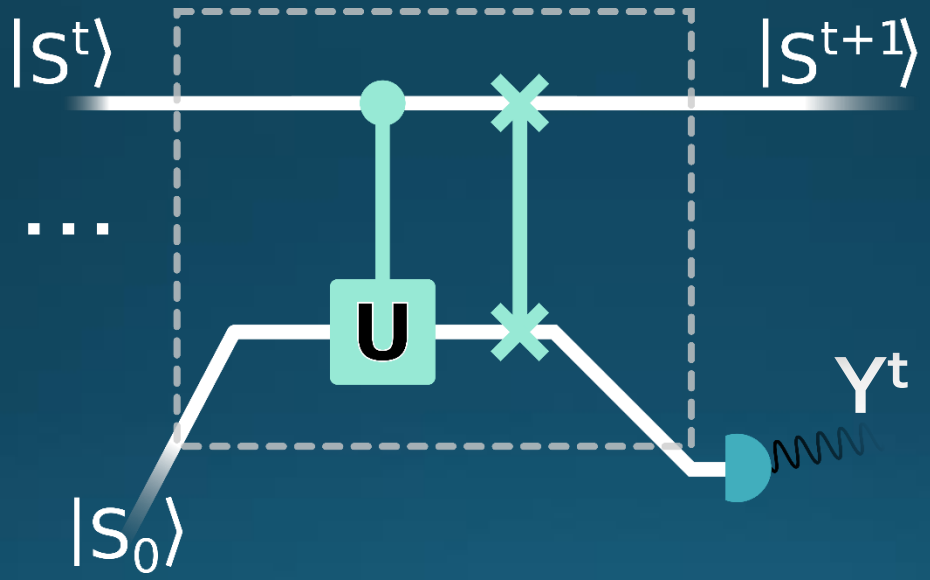
$$|S_1\rangle = \sqrt{p_{11}}\,|1\rangle + \sqrt{p_{21}}\,|2\rangle + \sqrt{p_{31}}\,|3\rangle + \sqrt{p_{41}}\,|4\rangle$$

Measurement of $|S_1\rangle$ in the 'which site?' basis collapses state onto one of the four possible sites with the correct probability.

Gu et al. Nature Communications (2012)

Quantum simulator

When supplied with appropriate state $|S^t\rangle$, the statistics for Y^t, and the next state may be generated by the following circuit:

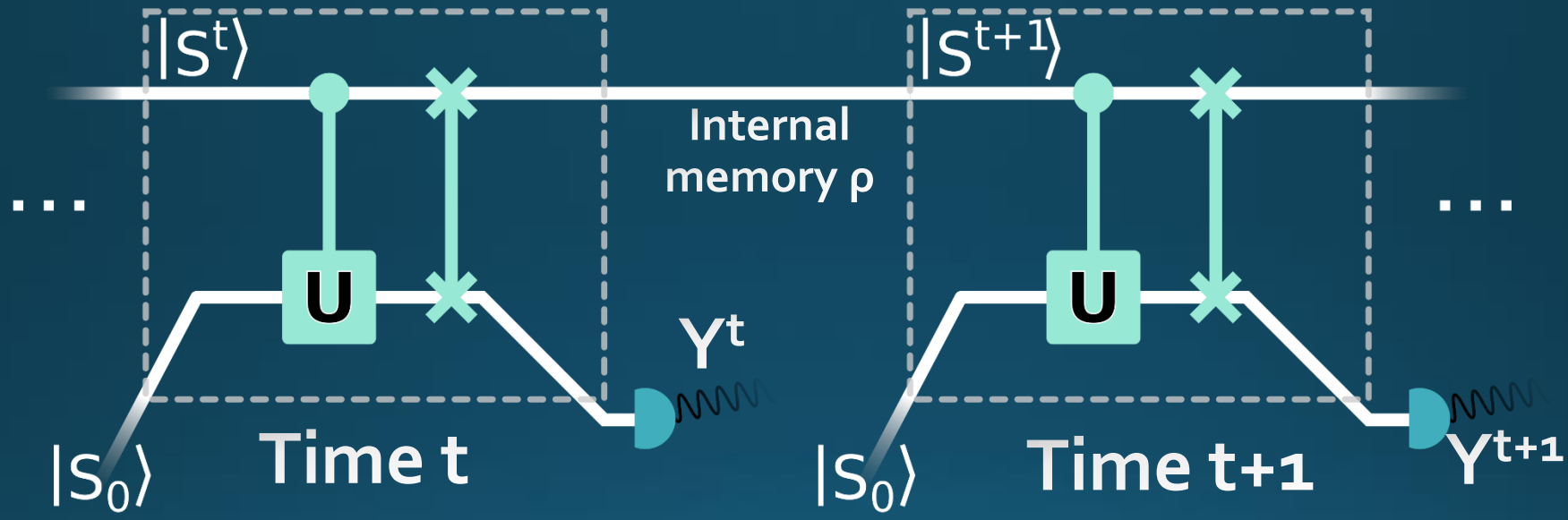


where U is a controlled unitary that for each *i* rotates initial state $|S_o\rangle$ to state $|S_i\rangle$ when the control bit is in state $|i\rangle$.

Gu et al. Nature Communications (2012)

Quantum simulator

Internal memory ρ persists between time-steps:



Steady-state of internal memory is $\rho = \frac{1}{2^n}\sum_{i=1}^{2^n}|S_i><S_i|$.

Memory requirement quantified by *von Neumann* entropy:

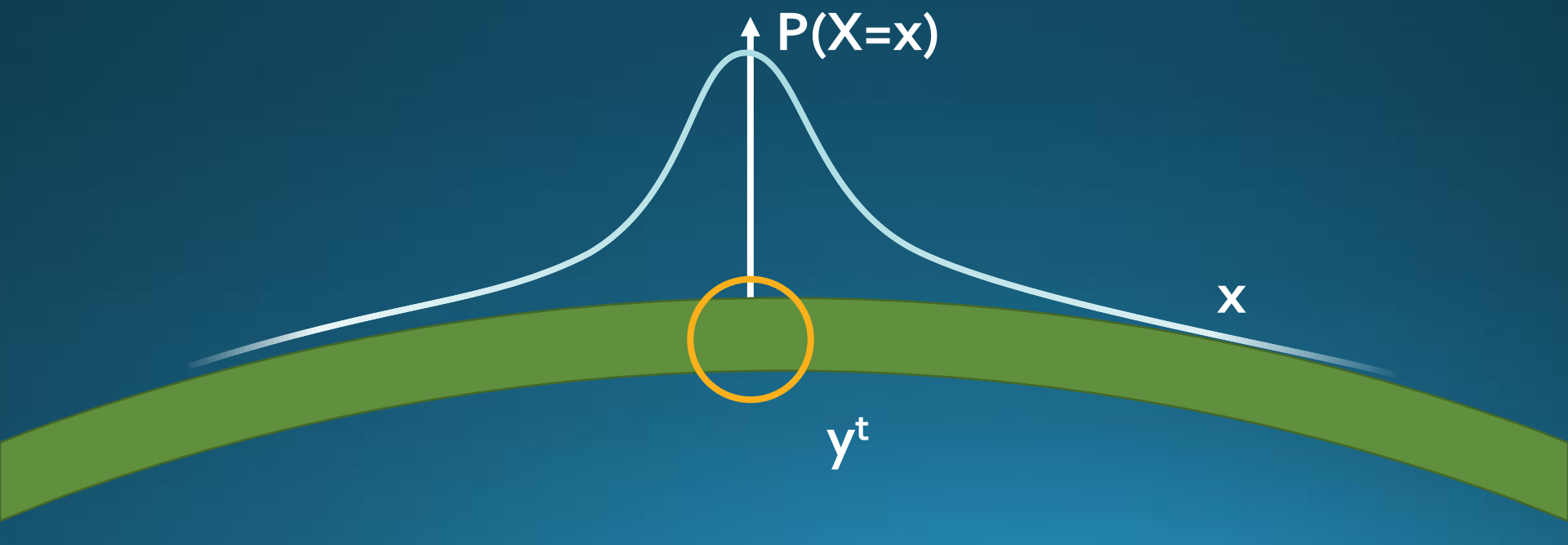
$$H = -Tr\rho \log \rho$$

Gu et al. Nature Communications (2012)

Example: Gaussian noise

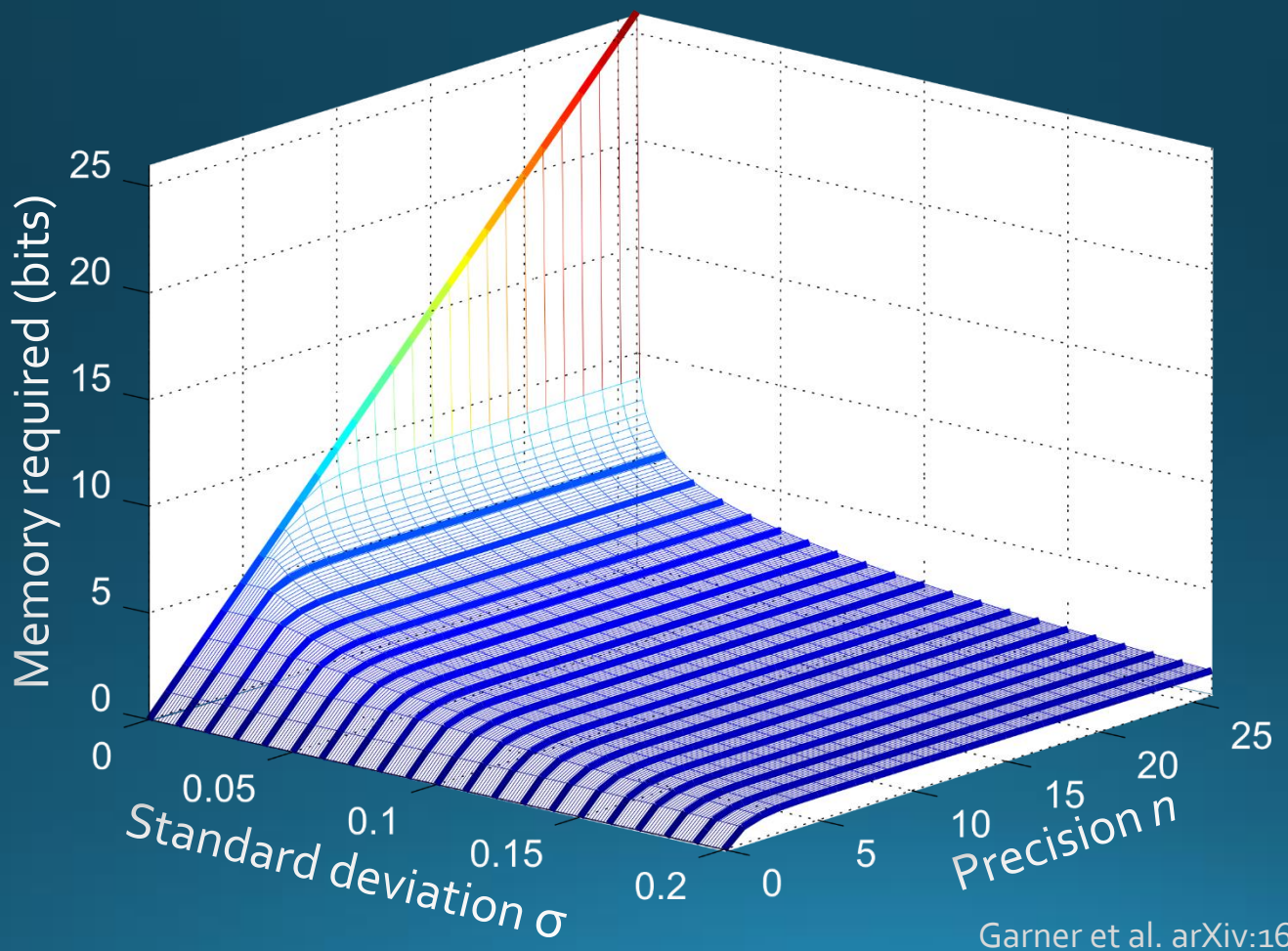
Shift function X describes Gaussian noise with standard deviation σ:

$$P(X = x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



Example: Gaussian noise

Entropy cost vs. precision:

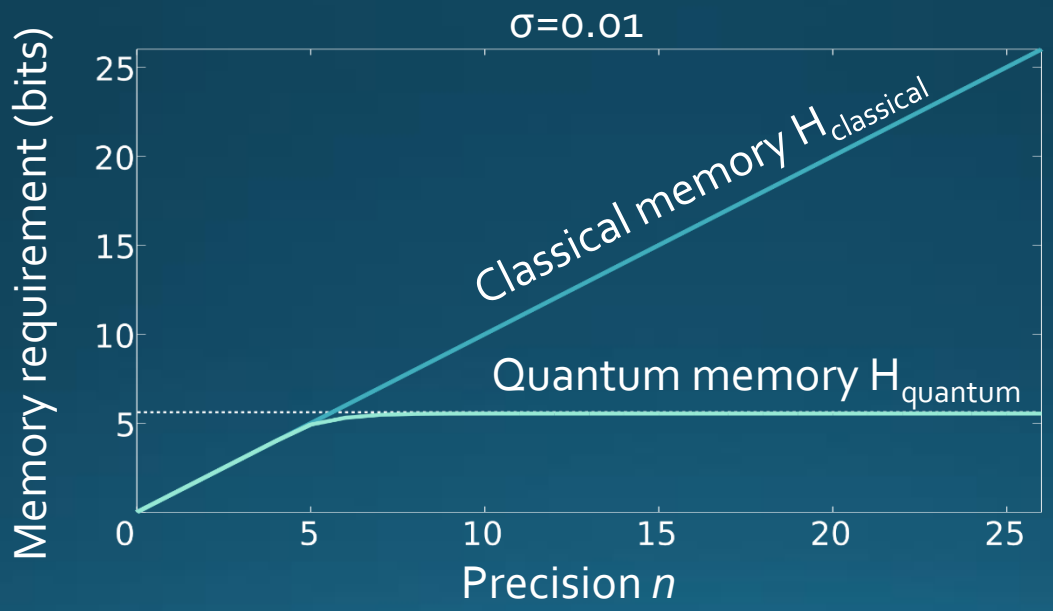


Garner et al. arXiv:1609.04408 (2016)

Example: Gaussian noise

Analytical bound for standard deviation $0 < \sigma \ll 1$:

$$\lim_{N\to\infty} H_{\text{quantum}} \leq \frac{1}{2\ln 2} - \left(1 + 4\sqrt{2}\sigma\right)\log_2 \sigma$$



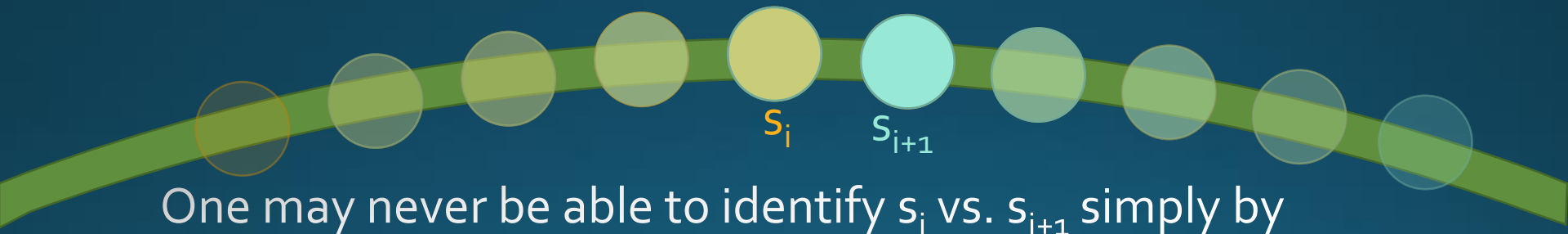
For small σ, leading term is $-\log_2 \sigma$

Origin of quantum advantage

As precision increases, the statistical future of neighbouring sites s_i, s_{i+1} becomes increasingly similar:

$$P\left(Y^{t+1}Y^{t+2}\ldots\middle|S^t = s_i\right) \approx P\left(Y^{t+1}Y^{t+2}\ldots\middle|S^t = s_{i+1}\right)$$

but never actually converge for any finite precision *n*.



One may never be able to identify s_i vs. s_{i+1} simply by looking at future statistics $Y^{t+1}Y^{t+2}\ldots$ – the process exhibits unboundedly large ***crypticity***.

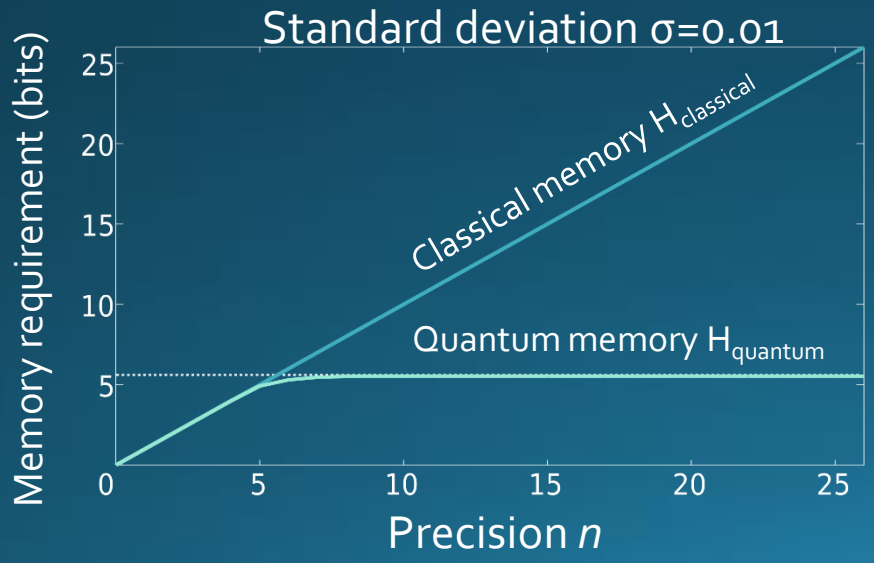
But, memory states s_i and s_{i+1} remain distinguishable! Hence, increasing precision **always** increases classical memory cost.

Origin of quantum advantage

Increasing precision **always** increases classical memory cost.

As precision increases quantum states $|S_i\rangle$ and $|S_{i+1}\rangle$ increasingly overlap: $\langle S_i|S_{i+1}\rangle \rightarrow 1$

Storing increasingly overlapping states adds a diminishing contribution to the memory's von Neumann entropy



Ultimately, precision can be increased with no further quantum memory cost.

Observer-dependent complexity

The classical memory cost of simulation corresponds to its **statistical complexity**[1].

Processes involving the stochastic dynamics of real numbers could be of very high complexity.

An observer with quantum information processing power could find such a process to be *much* simpler[2].

Quantum simulators can simulate to arbitrarily high precision at fixed finite memory cost:

1. Crutchfield & Young, PRL (1989)
2. Suen et al. arXiv:1511.05738 (2015); Aghamohammadi et al. arXiv:1602.08646 (2016)

Conclusions & outlook

Conclusions

- Simulation of stochastic processes on classical digital computers requires a trade-off between precision and memory required.
- Quantum simulators can simulate to arbitrarily high precision at a *finite memory cost.*
- Quantum observers may find some processes simple, even if they exhibit unboundedly high classical complexity.

Outlook

- These results probably generalize to any sort of simulation with *diffusive randomness*. (For example: non-Markovian or non-symmetric processes).
- This advantage may generalize to general *Monte Carlo* simulations, which could be run on a quantum processor to arbitrary precision, requiring a finite fixed memory cost.

Garner et al. arXiv:1609.04408

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